Final exam — Several Variable Calculus (Limited Version)

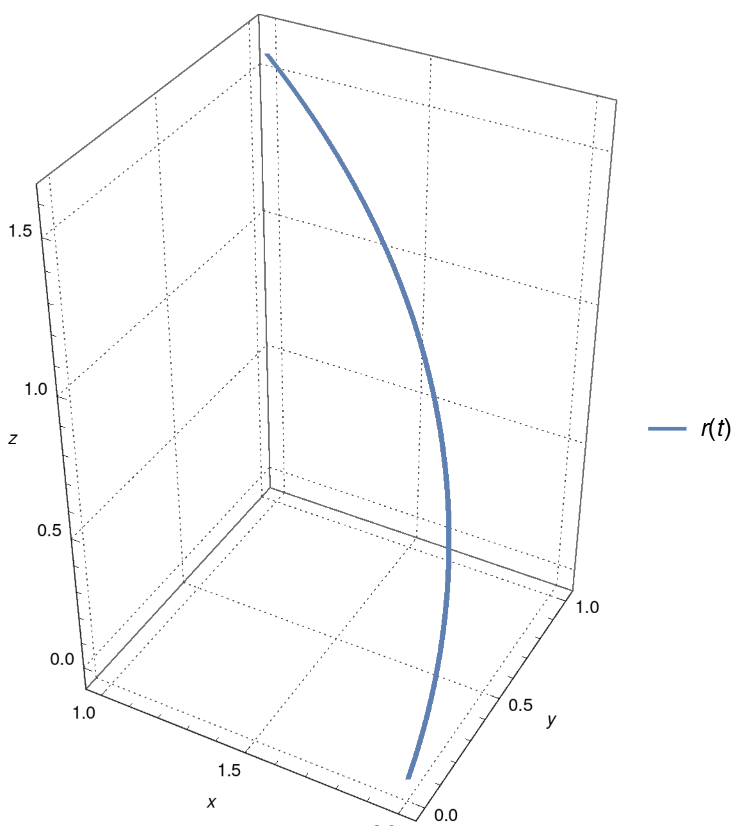
October 2019

1. Length of a curve

Consider the parametrization of a curve $\boldsymbol{r}(t) = (\cos(t) + 1, \sin(t), t)$ with $0 \leq t \leq \frac{\pi}{2}$.

- a) Draw the curve. (1p)
- b) Compute the velocity and the acceleration. (1p)
- c) Compute the length of the curve between $t = 0$ and $t = \frac{\pi}{2}$. (3p)

a) Picture



b) Velocity and acceleration

$r'(t) = (-\sin(t), \cos(t), 1)$

$r''(t) = (-\cos(t), -\sin(t), 0)$

c) Length

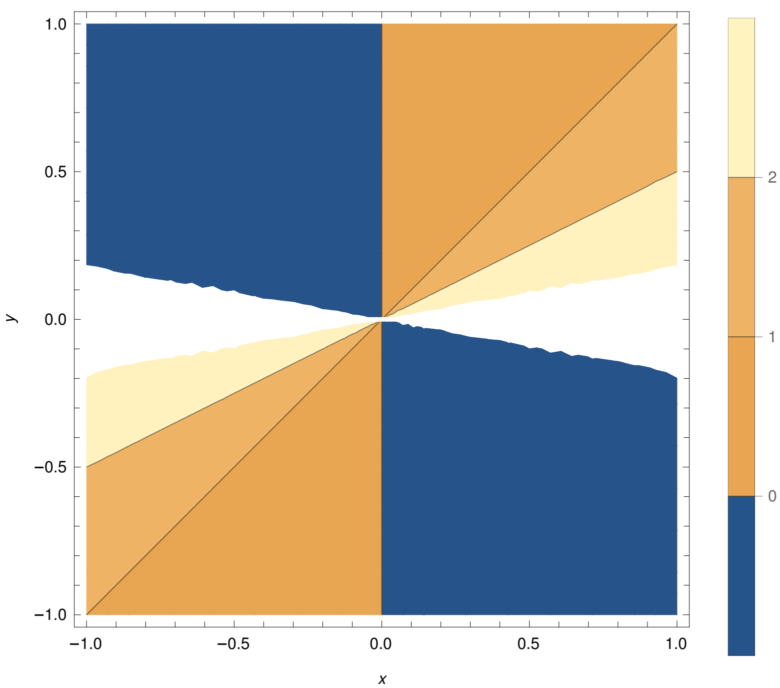
$$\int_0^{\frac{\pi}{2}} \sqrt{2}\, dt = \frac{\pi}{\sqrt{2}}$$

2. Function of several variables

Consider the function $f(x, y) = \frac{x}{y}$.

- a) Draw the 3 level curves $f(x, y) = 0$, $f(x, y) = 1$ and $f(x, y) = 2$. (2p)
- b) Does the function has a limit at (0, 0)? Justify your answer. (3p)

a) The level curves



b) limit

No. Because $\lim_{x \to 0} f(x, x) = 1$ and $\lim_{y \to 0} f(0, y) = 0$.

3. Chain rule

Let $f(x, y)$ be a function from $\mathbb{R}^2 \to \mathbb{R}$. Let $g(r, \theta) = f(r\cos(\theta), r\sin(\theta))$.

- a) Find $\frac{\partial}{\partial\theta} g(r, \theta)$. (3p)
- b) Assume that $\frac{\partial f}{\partial x}(0, 1) = 1$ and $\frac{\partial f}{\partial y}(0, 1) = 0$. Compute $\frac{\partial}{\partial\theta} g\left(1, \frac{\pi}{2}\right)$. (2p)

a)

$$-\frac{\partial f}{\partial x} r \operatorname{Sin}[\theta] + \frac{\partial f}{\partial y} r \operatorname{Cos}[\theta]$$

b)

-1

4. Taylor polynomial and extreme points

Let $f(x, y) = x + x^2 - y^2$.

- a) Find the only critical point of $f(x, y)$. (2p)
- b) Is this critical point a local minimal point, local maximal point, or a saddle point? (2p)
- c) Find the second degree Taylor polynomial of $f(x, y)$ at this point. (1p)

a)

The critical points are

$$\left\{\left\{-\frac{2}{3}, 0\right\}, \{0, 0\}\right\}$$

b)

The hessian matrix is

$$\begin{pmatrix} 2+6x & 0 \\ 0 & -2 \end{pmatrix}$$

At the two critical points, this matrix is

$$\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

So the D_1 and D_2 fro the two points are

{{-2, 4}, {2, -4}}

Thus the first is a local maxima and the latter is a saddle point.

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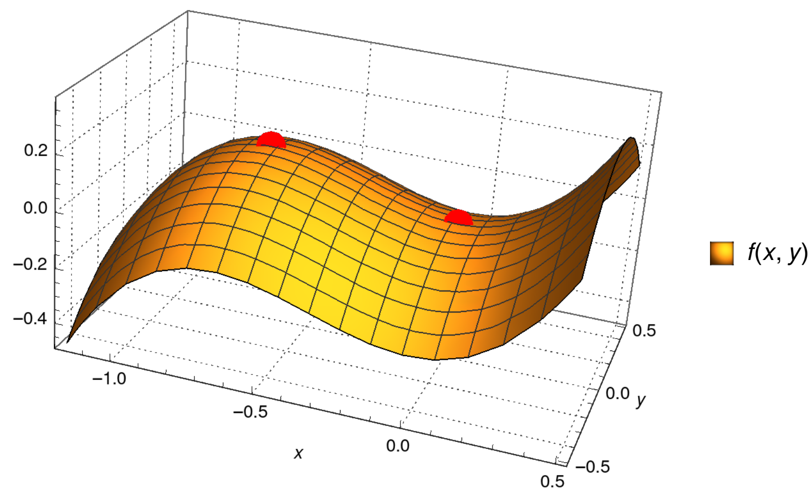
$\left\{\left\{-\frac{2}{3}, 0\right\}, \{0, 0\}\right\}$



$\left\{\left\{-\frac{2}{3}, 0, \frac{4}{27}\right\}, \{0, 0, 0\}\right\}$

c) Taylor polynomial

At these two points, the corresponding Taylor polynomials are

$\frac{4}{27} - \left(\frac{2}{3} + x\right)^2 - y^2$

$x^2 - y^2$

5. Optimization

Let $f(x, y, z) = x + 2y - 3z$. Find the maximal value and minimal value of $f(x, y, z)$ on the surface $g(x, y, z) = 0$ where $g(x, y, z) = x^2 + 4y^2 + 9z^2 - 3$.

Solution

Let

$l[x, y, z, \lambda] == x + 2y - 3z + \left(-3 + x^2 + 4y^2 + 9z^2\right)\lambda$

(1 points for writing this done).

The critical points of this function are (3 points)

$\left\{\left\{x \to -1, y \to -\frac{1}{2}, z \to \frac{1}{3}, \lambda \to \frac{1}{2}\right\}, \left\{x \to 1, y \to \frac{1}{2}, z \to -\frac{1}{3}, \lambda \to -\frac{1}{2}\right\}\right\}$

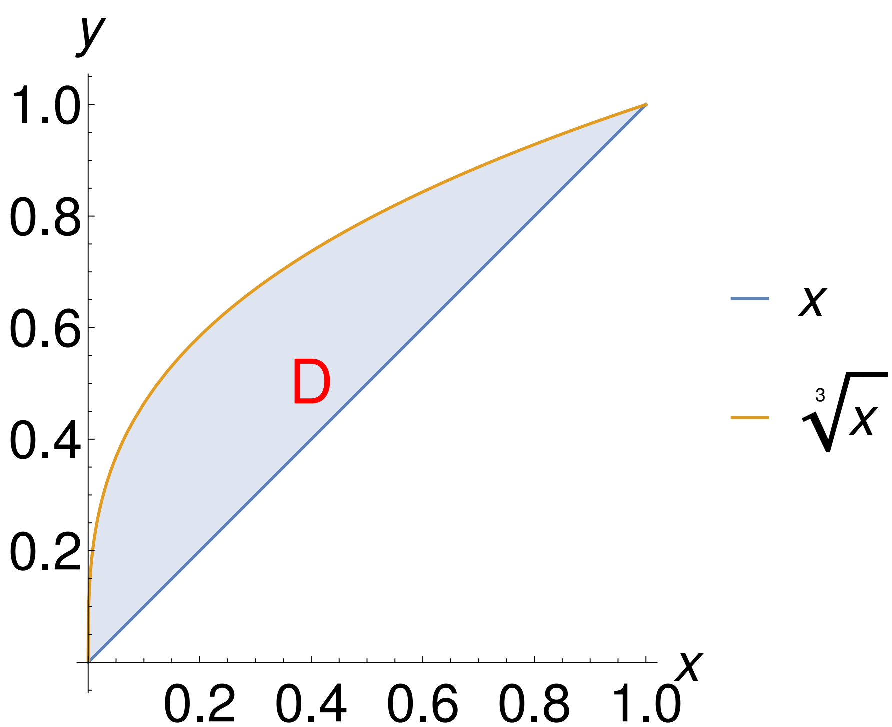
At these two points, $f(x, y, z)$ are

{3, -3}

So the maximal value is 3 and the minimal value is −3. (2 points)

6. Double integral

Compute the double integral $\iint_D (1 - y^2)\, dA$ over the domain D as shown in the picture. Hint: Integrating x first is slightly easier. (5p)



Solution

$$\int_0^1 \int_{y^3}^{y} (1 - y^2)\, dx\, dy$$

$$\frac{1}{6}$$

7. Vector field

Let $\boldsymbol{F}(x, y, z) = (y, x, z^2)$. The vector field $\boldsymbol{F}$ is conservative.

- a) Find the potential function $\phi(x, y, z)$ of $\boldsymbol{F}$. (3p)

- b) Compute the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ along the line from (0, 0, 0) to (1, 1, 1). (2p)

a)

$$\phi(x, y, z) = x\,y + \frac{z^3}{3}$$

b)

This is simply

$$\phi(1, 1, 1) - \phi(0, 0, 0) = \frac{4}{3}$$

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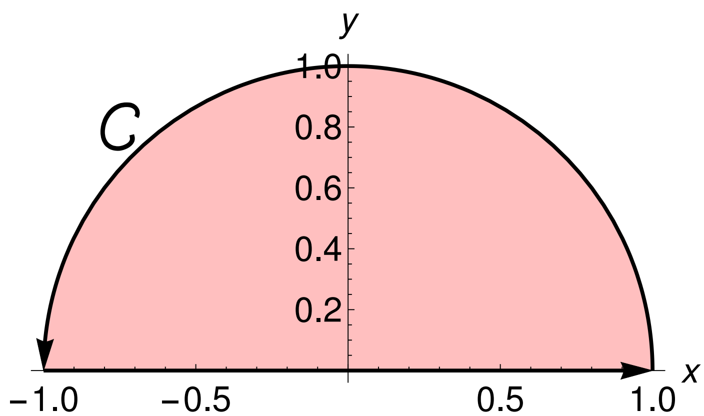
8. Green’s theorem

Let $P(x, y) = x^2 - 3\,x\,y + e^x + \sin(x)$ and $Q(x, y) = 2\,x\,y - y^2 + \log(y) - \cos(y)$. Let $\boldsymbol{F}(x, y) = (P, Q)$. Compute $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where C is boundary of half-disk $x^2 + y^2 \leq 1$ and $y \geq 0$, oriented counterclockwise.



Solution

$$\int_0^1 \int_0^\pi r\,(3\,r\,\text{Cos}[\theta] + 2\,r\,\text{Sin}[\theta])\,d\theta\,d r == \frac{4}{3}$$