Lecture 01 – Vectors and Coordinate Geometry in 3-Space

Several Variable Calculus, 1MA017

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Summary

Please watch these videos before the lectures: 1 and 2.

What we will talk about today

- 10.1: Analytical geometry in Three Dimensions
- 10.5: Quadric Surfaces
- 10.6: Cylindrical and spherical coordinates

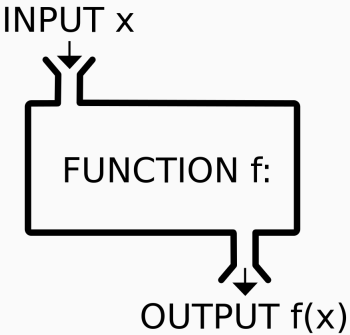
What you should already know (review if you forgot)

- P.2: Cartesian Coordinates in the Plane
- P.3: Graphs of Quadratic Equations
- 8.1: Conics
- 8.5: Polar Coordinates and Polar Curves

Introduction of the course

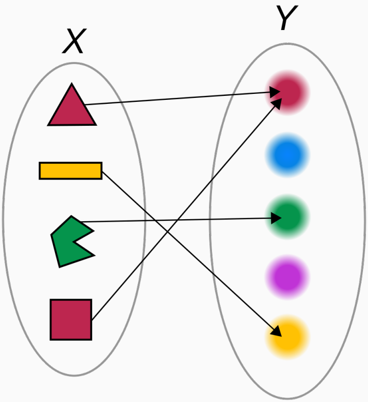
Function

A function is a “machine” or “black box” that maps input into output.



Function

More precisely, a function is a relation between sets that associates to every element of a first set exactly one element of the second set.



Different types of functions

Calculus studies 4 types of functions

1. real-valued functions of a single real variable,
2. vector-valued functions of a single real variable,
3. real-valued functions of a real vector variable, and
4. vector-valued functions of a real vector variable.

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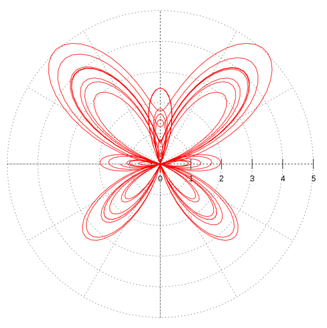
Single Variable Calculus covers the first one.

Several Variable Calculus covers the rest. We will study their limits, continuity, derivatives, integrals, etc.

Example — vector-valued functions of a single real variable

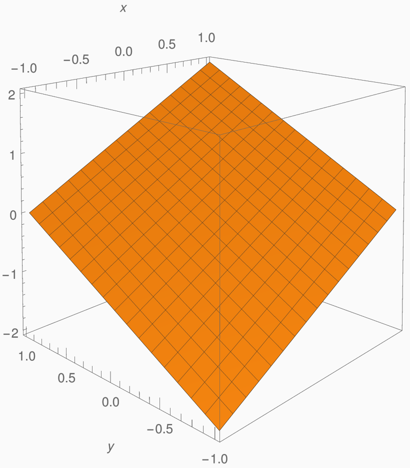
The butterfly curve is defined by the function $t \mapsto (x, y)$ where

$$x = \sin(t)\left(e^{\cos(t)} - 2\cos(4t) - \sin^5\left(\frac{t}{12}\right)\right)$$
$$y = \cos(t)\left(e^{\cos(t)} - 2\cos(4t) - \sin^5\left(\frac{t}{12}\right)\right)$$



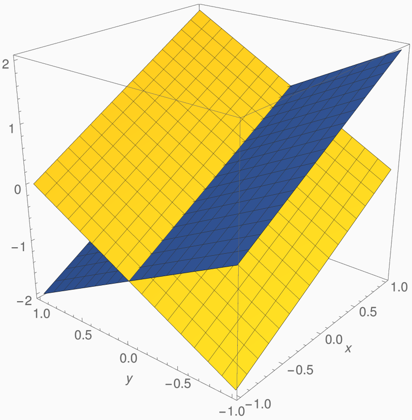
Example — real-valued functions of a real vector variable

This plane is defined by the function $(x, y) \mapsto x + y$.



Example — vector-valued functions of a real vector variable

The two planes are defined by the function $(x, y) \mapsto (x + y, x - y)$.



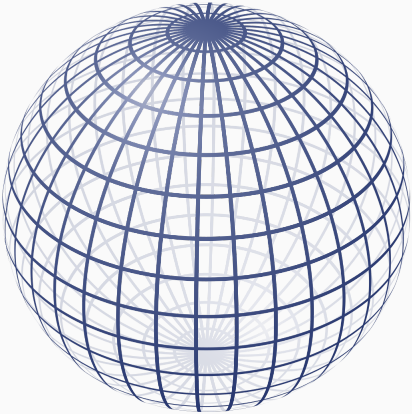
Applications

Several Variable Calculus is widely used in

- biology (population evolution)
- physics (general relativity)
- thermodynamics (air conditioner)
- robotics (motion control)
- aerospace (spaceship trajectory)
- machine learning (algorithm analysis)

Application — The surface of a sphere

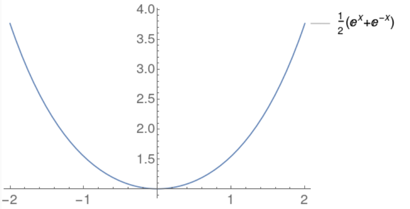
The surface of a sphere of radius r is $4\pi r^2$. Where does this number come from?



Application — Catenary bridges

The curve of a suspension bridge is a function of the form $x \mapsto \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$.

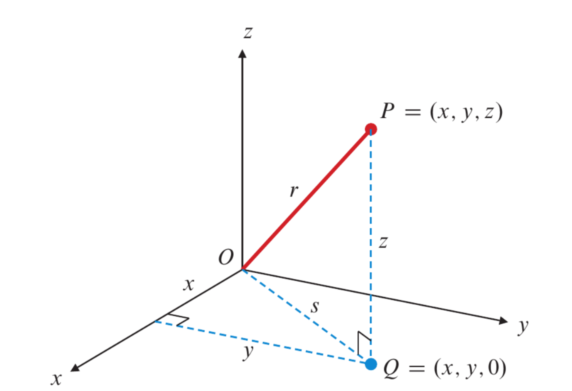
Given a, how can we compute the length of the bridge?



10.1 Analytic Geometry in Three Dimensions

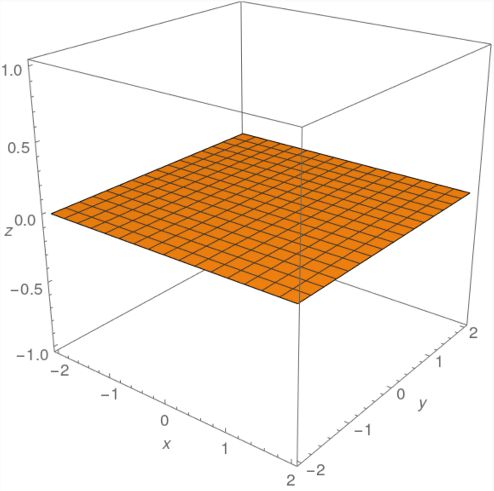
A point in $\mathbb{R}^3$

$\mathbf{P} = (x, y, z) \in \mathbb{R}^3$ can be seen as the coordinates of a point P in the 3-dimensional space in which we live.



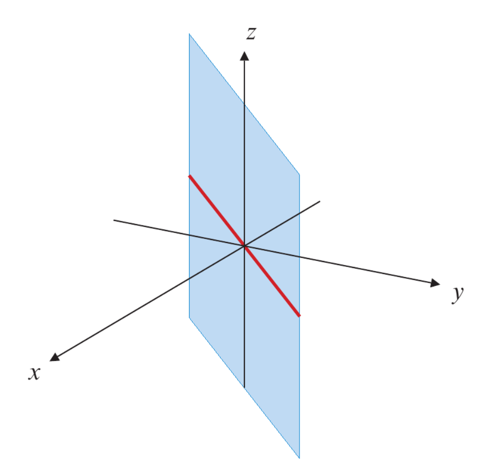
Various geometric objects can be seen as subsets of $\mathbb{R}^3$.

Planes in $\mathbb{R}^3$



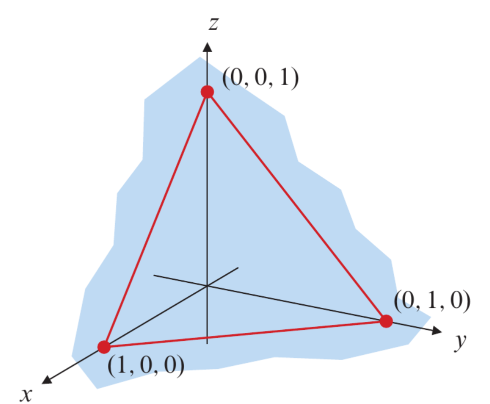
$$z = 0$$

Planes in $\mathbb{R}^3$



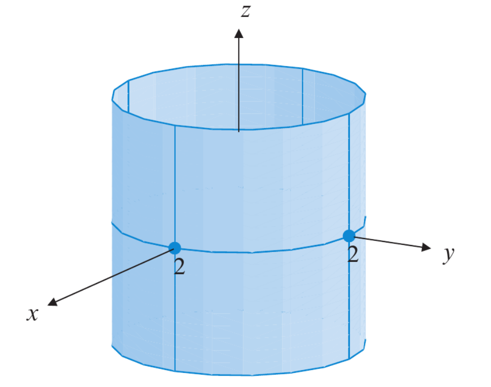
$$x = y$$

Planes in $\mathbb{R}^3$



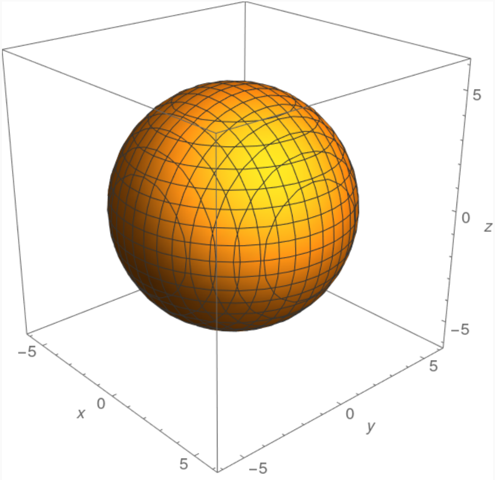
$$x + y + z = 1$$

Surface in $\mathbb{R}^3$

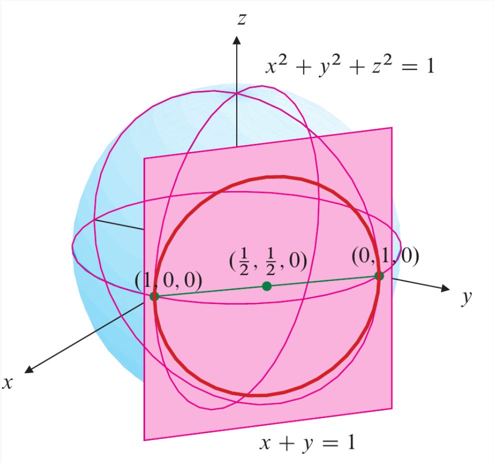


$$x^2 + y^2 = 1$$

Surface in $\mathbb{R}^3$



$$x^2 + y^2 + z^2 = 25$$



Intersection of $x^2 + y^2 + z^2 = 1$ and $x + y = 1$

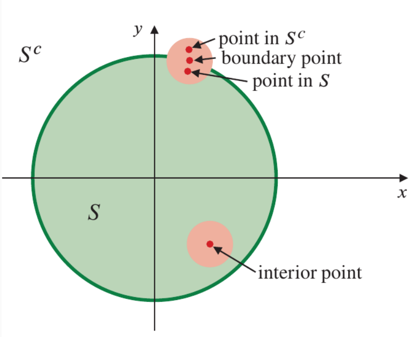
Topology in $\mathbb{R}^n$ 😕

A neighbourhood of a point a:

$$B_r(a) = \{x : |x - a| < r\}$$

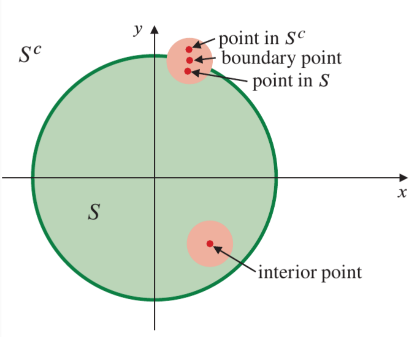
A point a is said to be

- an interior point of S if a has an neighbourhood in S.
- an exterior point of S if a has an neighbourhood outside S.
- a boundary point of S if every neighbourhood of a both contains points in and out S.



Topology in $\mathbb{R}^n$ 😕

- A set S is said to be open if every point $a \in S$ has an neighbourhood that lies entirely in S (all points in S are then interior points)
- A set S is said to be closed if its complement is an open set (all boundary points belong to S)



Example: Topology in $\mathbb{R}^2$

Which of these sets are open/closed/none? Can you determine boundary points?

- $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.
- $B = \{(x, y) \in \mathbb{R}^2 : x < 1\}$.

Quiz: Topology in $\mathbb{R}^2$

Which of these sets are open/closed/none? Can you determine boundary points?

- $C = \{(x, y) \in \mathbb{R}^2 : y = x^2\}$.
- $D = \{(x, y) \in \mathbb{R}^2 : y > x^2, y \leq x + 1\}$.

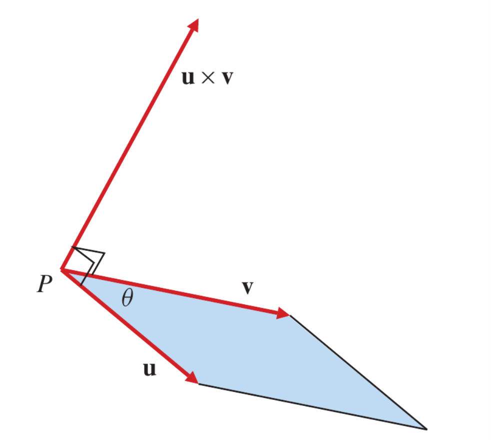
Quick Review — 10.2, 10.3 Vectors and Cross Product

Fundamental concepts

- Vectors
- Inner/Dot product $\mathbf{u} \cdot \mathbf{v} = v_1 u_1 + v_2 u_2 \ldots v_n u_n$.
- Norm/length/absolute value $|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$.
- Distance — $|\overrightarrow{\mathbf{P}_1\mathbf{P}_2}| = |\mathbf{P}_1 - \mathbf{P}_2|$.
- Orthogonal — $\mathbf{u} \cdot \mathbf{v} = 0$.

Cross Product

$\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ and has length equal to the area of the blue shaded parallelogram.



Other things you should know from linear algebra

How to

- Determine if the point $(1, 2, 3)$ lies in the plane $z = 2x - 3y + 4$
- Find a normal vector for the plane $z = 2x - 3y + 4$
- Determines if the vectors $(1, 2, 3)$ and $(-3, -2, -1)$ are orthogonal
- Find a vector that is orthogonal to both (1, 2, 0) and (1, -1, 2)

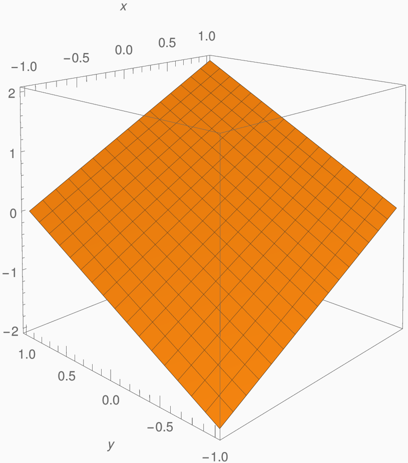
Quick Review — 10.4 Planes and Lines

Planes in $\mathbb{R}^3$

The equation

$$ax + by + cz = 0$$

defines a plane in $\mathbb{R}^3$



Lines in $\mathbb{R}^3$

Given a point $\mathbf{r}_0 = (x_0, y_0, z_0)$ and a vector $\mathbf{v} = (a, b, c)$ in $\mathbb{R}^3$, there is unique line that goes through $\mathbf{r}_0$ and is parallel to $\mathbf{v}$.

Lines in $\mathbb{R}^3$

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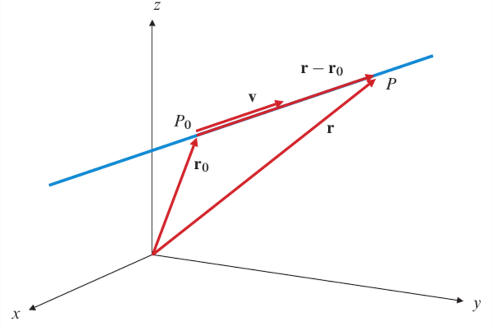
Any point $\mathbf{r}$ on this line satisfies $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$



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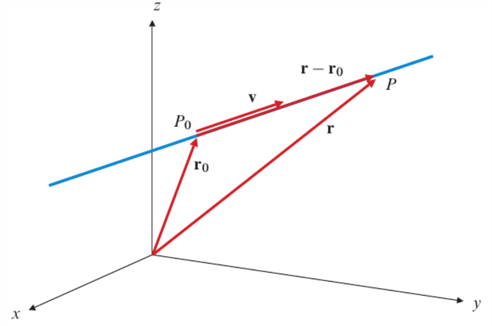


which is equivalent to

$$\{(x, y, z) \in \mathbb{R}^3 \mid x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct.\}$$

10.5 Quadric Surface

Review: Quadratic curves in $\mathbb{R}^2$ (P3 and 8.1)

- Parabola

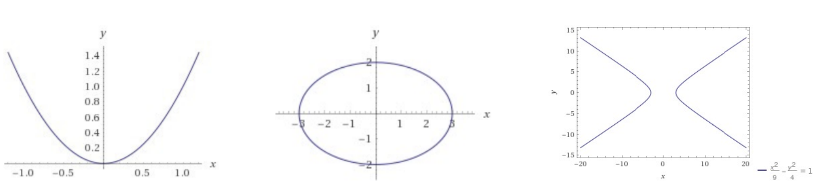
$$y = x^2$$

- Ellipse

$$x^2/9 + y^2/4 = 1$$

- Hyperbola

$$x^2/9 - y^2/4 = 1$$



Quadric Surfaces in $\mathbb{R}^3$

The most general second-degree equation in three variables is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz = J.$$

If the equation factorizes,

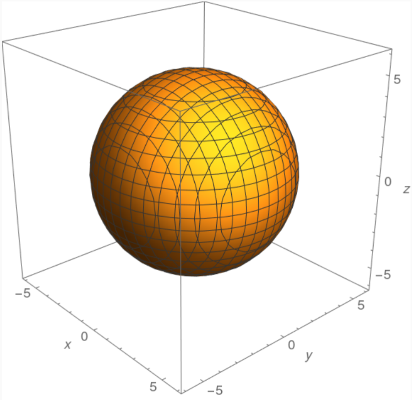
$$(A_1x + B_1y + C_1z - D_1)(A_2x + B_2y + C_2z - D_2) = 0,$$

then it's just two planes. Otherwise, the equations defines a quadric surface.

Sphere

A sphere is defined by the quadratic equation

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

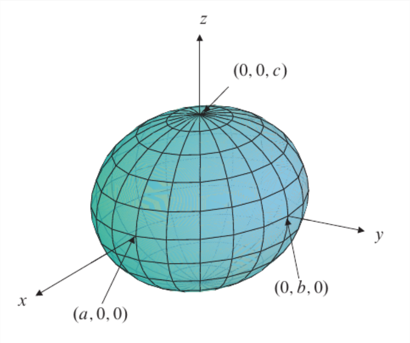


Sphere

Ellipsoid

An ellipsoid is defined by the quadratic equation

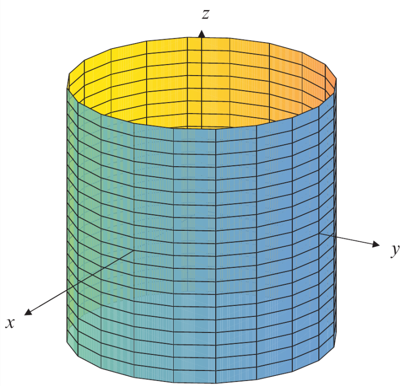
$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}+\frac{(z-z_0)^2}{c^2}=1$$



Ellipsoid

Cylinder

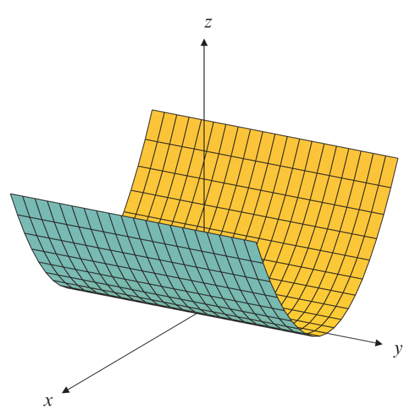
Examples of cylinders include



$$x^2 + y^2 = a^2$$

Cylinder

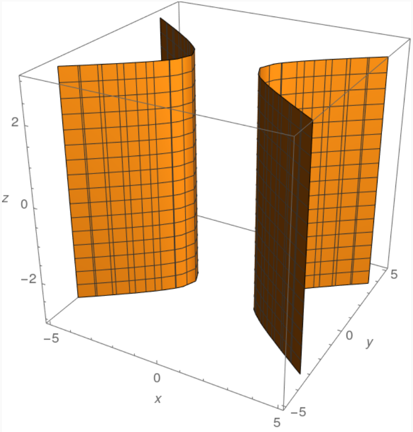
Examples of cylinders include



$$z = x^2$$

Cylinder

Examples of cylinders include

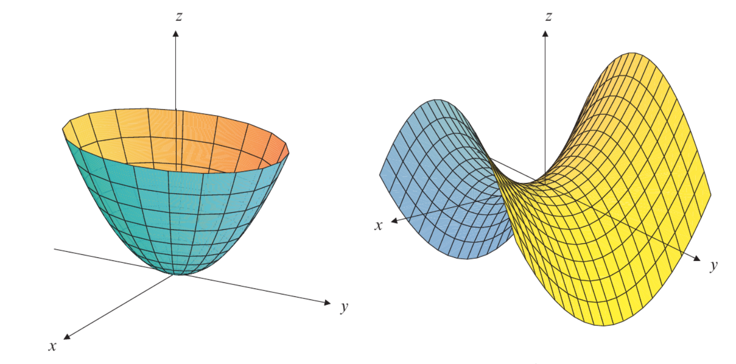


$$\frac{x^2}{2} - y^2 = 1$$

Paraboloid

A paraboloid is defined by

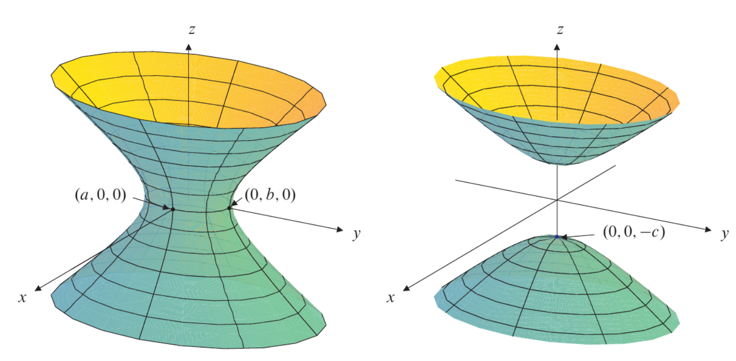
$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \qquad \text{or} \qquad z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$



Hyperboloid

A hyperboloid is defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad \text{or} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$



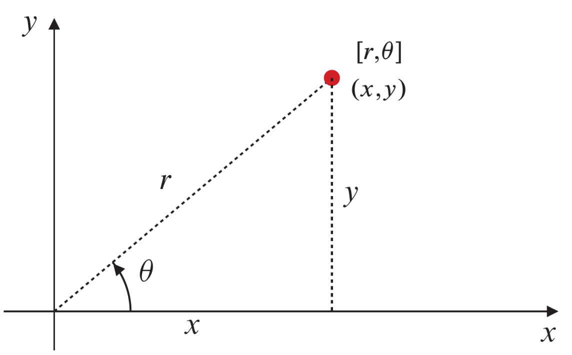
10.6 Coordinate systems

Polar system (8.5)

Conversion formula

$$x = r\cos\theta \qquad\qquad x^2 + y^2 = r^2$$

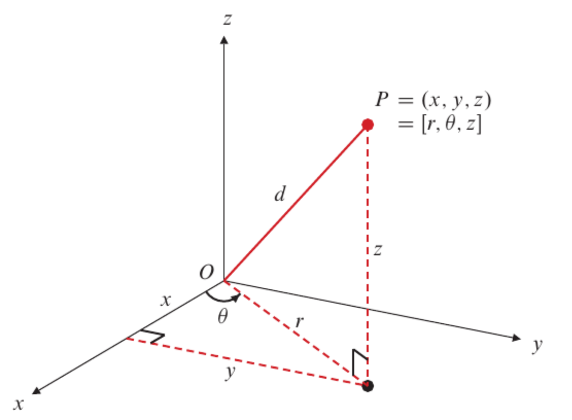
$$y = r\sin\theta \qquad\qquad \tan\theta = \frac{y}{x}$$



Cylindrical Coordinates

Conversion from cylindrical coordinates $[r, \theta, z]$

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$



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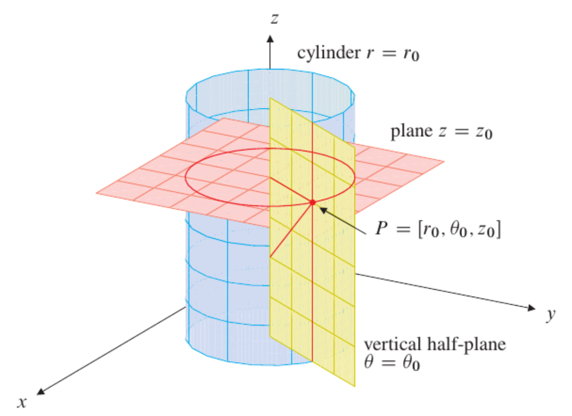
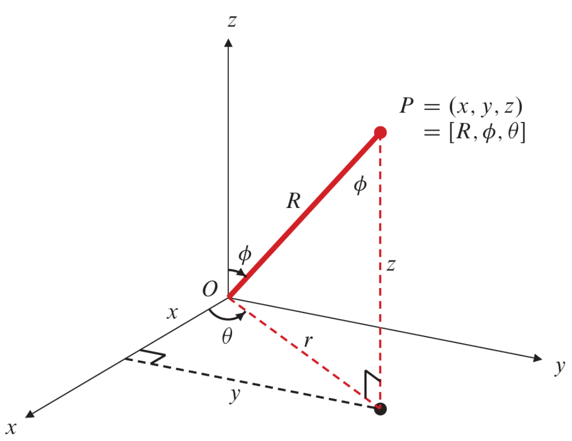


Spherical Coordinates

Conversion from spherical coordinates $[R, \varphi, \theta]$

$$x = R\sin\varphi\cos\theta, \qquad y = R\sin\varphi\sin\theta, \qquad z = R\cos\varphi.$$



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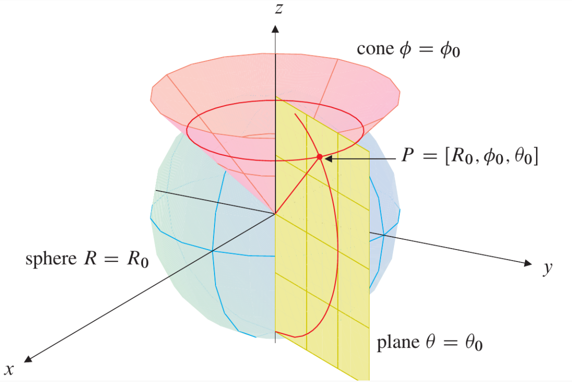
$$x = R\sin\varphi\cos\theta, \qquad y = R\sin\varphi\sin\theta, \qquad z = R\cos\varphi.$$



Quiz

Describe the following areas in polar coordinates

- $x^2 + y^2 < 1, x > 0.$
- $1 \leq x^2 + y^2 \leq 2, y \geq |x|.$
- $x^2 + y^2 \leq 1, x \leq |y|, y > 0.$

Quiz

Describe the following areas in spherical coordinates

$$x^2 + y^2 + z^2 \leq 1, \qquad x \geq 0, \qquad y \geq 0, \qquad z \geq 0.$$