Lecture 02 – Vector Functions and Curves

Several Variable Calculus, 1MA017

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Summary

Please watch these videos before the lecture: 3

What we will talk about today

- 11.1: Vector Functions of One Variable
- 11.3: Curves and Parametrizations

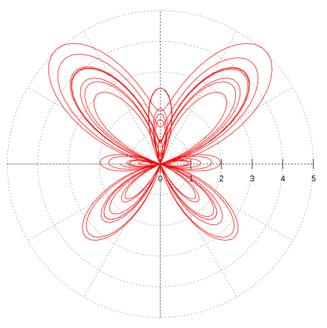
11.1 Vector function of one variable

Example — vector-valued functions of a single real variable

The butterfly curve is defined by the function $t \mapsto (x, y)$ where

$$x = \sin(t)\left(e^{\cos(t)} - 2\cos(4t) - \sin^5\left(\frac{t}{12}\right)\right)$$

$$y = \cos(t)\left(e^{\cos(t)} - 2\cos(4t) - \sin^5\left(\frac{t}{12}\right)\right)$$



Vector-valued function of one variable

We consider functions $\mathbf{r} : D \mapsto \mathbb{R}^n$, where D is an interval.

We write the components of $\mathbf{r}(t)$ as

$$\mathbf{r}(t) = \left(x_1(t), \dots, x_n(t)\right).$$

Usually in 3-space, we write

$$\mathbf{r}(t) = (x(t), y(t), z(t)).$$

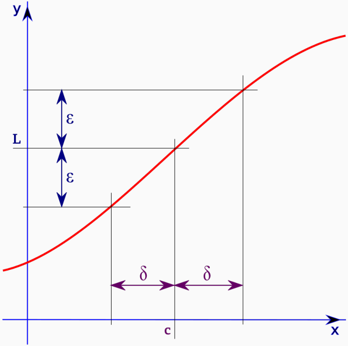
Concepts: Limit values, continuity, derivatives, integration — The same concepts from one-variable-calculus applied to each component.

Limits of functions

From $\mathbb{R}$ to $\mathbb{R}$

The notation $\lim_{x \to c} r(x) = L$ means that for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - c| < \delta \implies |r(x) - L| < \varepsilon$$



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From $\mathbb{R}$ to $\mathbb{R}^n$

The notation $\lim_{x\to c} \mathbf{r}(x) = \mathbf{L}$ means that for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - c| < \delta \implies |\mathbf{r}(x) - \mathbf{L}| < \varepsilon$$

Only difference: $\mathbf{r}(x) - \mathbf{L}$ is now a vector.

Continuity

From $\mathbb{R}$ to $\mathbb{R}$

The function $r(x)$ is continuous at a point $a \in D$ means

$$\lim_{x \to a} r(x) = r(a).$$

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The function $\mathbf{r}(x)$ is continuous at a point $a \in D$ means

$$\lim_{x \to a} \mathbf{r}(x) = \mathbf{r}(a).$$

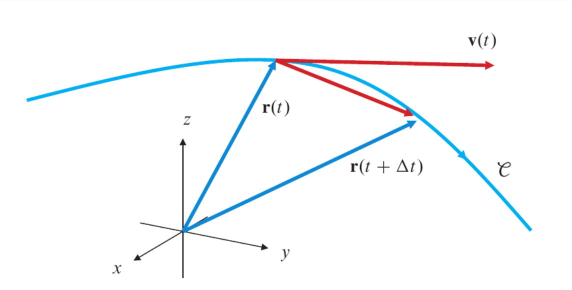
This is equivalent to that every component is continuous in one variable sense.

Derivative

The derivative of a function from $\mathbb{R}$ to $\mathbb{R}^n$ is defined by

$$\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if the limit exists. Note that the derivative is also a vector.



Interpretations of such functions

A continuous function $\mathbf{r}$ from an interval in $\mathbb{R}$ to $\mathbb{R}^n$ can be seen as a particle moves along the curve and $\mathbf{r}(t)$ indicates the particle's position at time t.

We can ask: What are the velocity and acceleration of the particle?

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We can ask: What are the velocity and acceleration of the particle?

Such a curve can also be seen as a geometric object.

We can ask: What is the tangent line of a curve at a certain point? How long is the curve?

Velocity

From the moving particle point of view, $\mathbf{r}'(a)$ is the velocity at the time a and the second derivative $\mathbf{r}''(a)$ is the acceleration.

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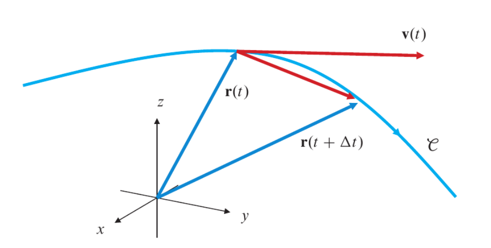
We often write $\mathbf{r}' = \mathbf{v}$ and $\mathbf{r}'' = \mathbf{a}$. Note that both velocity and acceleration are vectors.

The length of $\mathbf{v}$, i.e., $|\mathbf{v}|$, is called the speed and it is a number.



Tangent line

The tangent line of a curve $\mathbf{r}(t)$ at point t_0 is given by

$$\mathbf{T}(t) = \mathbf{r}(t_0) + \mathbf{v}(t_0)(t - t_0).$$

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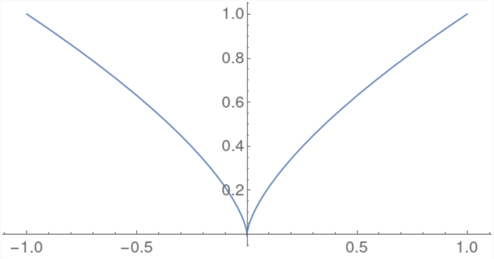
😲 This only works if the curve is "smooth", i.e., if the velocity (derivative) exists and is not zero.



$\mathbf{r}(t) = (t^3, t^2)$

Example

A particle moves along a curve in the xy-plane such that the position at the time t seconds after the start is

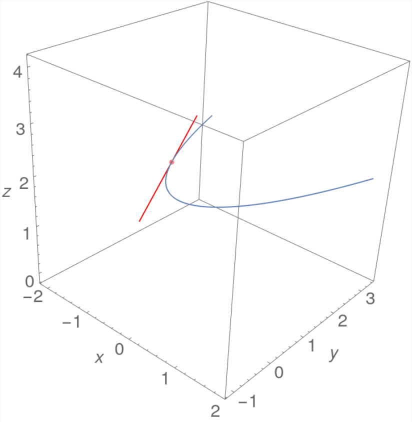
$$\mathbf{r}(t) = (2\cos \pi t, 3\sin \pi t), t \geq 0.$$

Draw the path of the particle and determine the direction of the tangent line at point $\mathbf{r}(2)$.

What is the name of this curve?

Quiz

Consider the function $\mathbf{r}(t) = (t, t^2, 2)$. Show that the point $(-1, 1, 2)$ is on the curve and find the tangent line of the curve at this point.



Differentiation rules

$$\text{(a)} \qquad \frac{d}{dt}\Big(\mathbf{u}(t) + \mathbf{v}(t)\Big) = \mathbf{u}'(t) + \mathbf{v}'(t)$$

$$\text{(b)} \qquad \frac{d}{dt}\Big(\lambda(t)\mathbf{u}(t)\Big) = \lambda'(t)\mathbf{u}(t) + \lambda(t)\mathbf{u}'(t)$$

$$\text{(c)} \qquad \frac{d}{dt}\Big(\mathbf{u}(t) \bullet \mathbf{v}(t)\Big) = \mathbf{u}'(t) \bullet \mathbf{v}(t) + \mathbf{u}(t) \bullet \mathbf{v}'(t)$$

$$\text{(d)} \qquad \frac{d}{dt}\Big(\mathbf{u}(t) \times \mathbf{v}(t)\Big) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$$

$$\text{(e)} \qquad \frac{d}{dt}\Big(\mathbf{u}\big(\lambda(t)\big)\Big) = \lambda'(t)\mathbf{u}'\big(\lambda(t)\big).$$

Also, at any point where $\mathbf{u}(t) \neq \mathbf{0}$,

$$\text{(f)} \qquad \frac{d}{dt}|\mathbf{u}(t)| = \frac{\mathbf{u}(t) \bullet \mathbf{u}'(t)}{|\mathbf{u}(t)|}.$$

Example: Product rules

EXAMPLE 7 If $\mathbf{u}$ is three times differentiable, calculate and simplify the triple product derivative

$$\frac{d}{dt}\left(\mathbf{u} \bullet \left(\frac{d\mathbf{u}}{dt} \times \frac{d^2\mathbf{u}}{dt^2}\right)\right).$$

Solution Using various versions of the Product Rule, we calculate

$$\begin{aligned}\frac{d}{dt}&\left(\mathbf{u} \bullet \left(\frac{d\mathbf{u}}{dt} \times \frac{d^2\mathbf{u}}{dt^2}\right)\right)\\ &= \frac{d\mathbf{u}}{dt} \bullet \left(\frac{d\mathbf{u}}{dt} \times \frac{d^2\mathbf{u}}{dt^2}\right) + \mathbf{u} \bullet \left(\frac{d^2\mathbf{u}}{dt^2} \times \frac{d^2\mathbf{u}}{dt^2}\right) + \mathbf{u} \bullet \left(\frac{d\mathbf{u}}{dt} \times \frac{d^3\mathbf{u}}{dt^3}\right)\\ &= 0 + 0 + \mathbf{u} \bullet \left(\frac{d\mathbf{u}}{dt} \times \frac{d^3\mathbf{u}}{dt^3}\right) = \mathbf{u} \bullet \left(\frac{d\mathbf{u}}{dt} \times \frac{d^3\mathbf{u}}{dt^3}\right).\end{aligned}$$

Question you may get in the exam

Let $r(t)$ describe the position of a particle in the xy-plane where it moves at a constant angular velocity ω radians per second in a circle with radius R around the origin.

- Write done the expression for $\mathbf{r}(t)$ if the particle is at the point $(R, 0)$ at time $t = 0$ seconds.
- Calculate the velocity $\mathbf{r}'(t)$ and the acceleration $\mathbf{r}''(t)$ of the particle.
- Draw a picture of the particle's trajectory and draw the velocity vector and acceleration vector at a time of your choice.

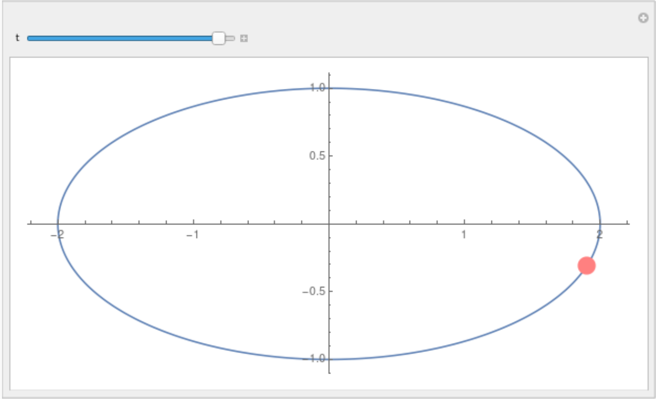
11.3 Curves and parametrizations

Parametrizations

Parametrization of curves – Define a curve in $\mathbb{R}^n$ as function

$$\mathbf{r}(t) = (x_1(t), \dots, x_n(t)).$$

Practice parametrizations of curves. It is important. 😲



$$r(t) = (2\cos(\pi t), \sin(\pi t))$$

One curve, many parametrizations

The same curve can have many parametrizations. For example,

$$\mathbf{r}(t) = (t, 2t), \qquad t \in [0, 1]$$

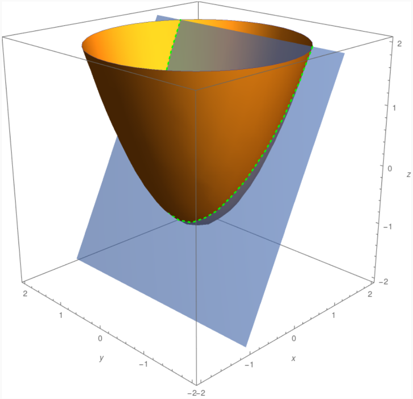
and

$$\mathbf{r}(t) = (\sqrt{100t}, 2\sqrt{100t}), \qquad t \in [0, 1/100]$$

give the same curve.

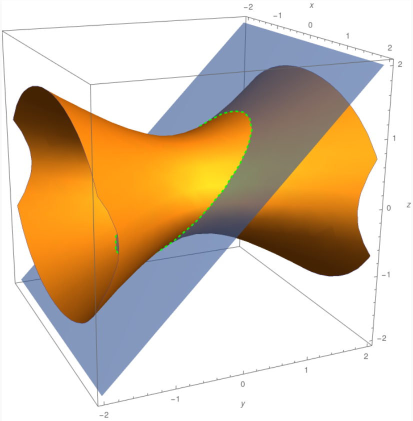
Example: Intersection of two surfaces

Determine a parametrization for the curve given by the intersection of the surface $z = x^2 + y^2 - 1$ and $z = 2x$.



Quiz: Intersection of two surfaces

Determine a parametrization for the curve given by the intersection of the area $x^2 - y^2 + 2z^2 = 1$ and $z = y$.



Choices:

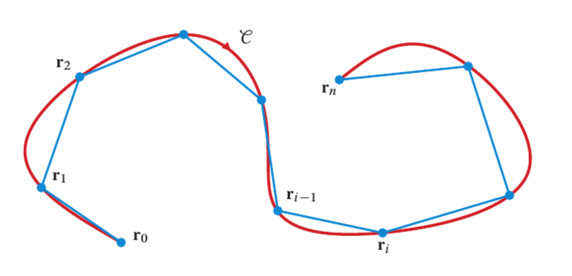
1. $(\sin(t), \cos(t), \sin(t))$
2. $(\sin(t), \cos(t), \cos(t))$
3. $(2\cos(t), 2\sin(t), t)$
4. $(\cos(\pi t), \sin(\pi t), \sin(\pi t))$
5. $(\cos(\pi t), \sin(\pi t), \pi t)$

The arc-length of a curve

Let $\Delta t_i = t_i - t_{i-1}$. Let $\Delta \mathbf{r}_i = \mathbf{r}_i - \mathbf{r}_{i-1}$. Then the length of the curve $\mathcal{C}$ is approximately

$$s_n = \sum_{i=1}^{n} |\Delta \mathbf{r}_i| = \sum_{i=1}^{n} \left| \frac{\Delta \mathbf{r}_i}{\Delta t_i} \right| \Delta t_i.$$

The arc-length of the curve the supremum of all such approximations.



The arc-length of a smooth curve

If $\mathbf{r}(t)$, where $a \leq t \leq b$, is a parametrization of a smooth curve, then the length of the curve equals

$$\int_a^b |\mathbf{r}'(t)| \, \mathrm{d}t$$

The special case $\mathbf{r}(t) = (t, f(t))$, this is equivalent to

$$\int_a^b \sqrt{1 + |f'(t)|^2} \mathrm{d}t$$