# Lecture 03 – Function of several variables

Several Variable Calculus, 1MA017


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# Summary

Please watch this video before the lecture: 4

Today we will talk about

- 11.3: Curves and Parametrizations
- 12.1: Functions of Several Variables

# 11.3 Curves and parametrizations

# Review: Arc-length

If $\mathbf{r}(t)$, where $a \leq t \leq b$, is a parametrization of a smooth curve, then the length of the curve equals

$$\int_a^b |\mathbf{r}'(t)| \, \mathrm{d}t$$

**Example**

Find the arc-length of the curve $\mathbf{r}(t) = (\cos t, \sin t)$ for $t \in [0, \pi]$.

# Arc-length parametrization

Given a curve $\mathcal{C}$, a natural choice of parametrization $\mathbf{r}(s)$ is to make $s$ representing the arc-length from an initial point $P_0$ on the curve. This is called the arc-length parametrization.

**Example**

Find the arc-length parametrization of $\mathbf{r}(t) = (t, t)$.

# Find the arc-length parametrization

Given any parametrization $\mathbf{r}(t)$, we can try find $t(s)$, the inverse of

$$s(t) = \int_{t_0}^{t} |\mathbf{r}'(\tau)| \, d\tau.$$

Then $\mathbf{r}(t(s))$ gives the arc-length parametrization with $s$ being the arc-length from $t_0$.

# Example

Calculate the length of the curve

$$\mathbf{r}(t) = (3t, 4t), \qquad 0 \leq t \leq 2.$$

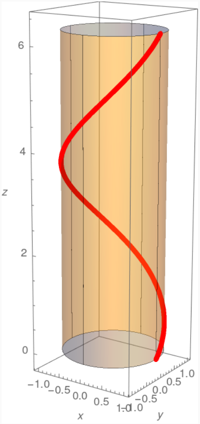
Also determine an arc length parametrization of the curve.

# Exam level question

Calculate the length of the spiral curve

$$\mathbf{r}(t) = (\cos t, \sin t, t), \qquad 0 \le t \le 2\pi.$$

Also determine an arc length parametrization of the curve.



# 12.1 Functions of Several Variables

# Functions from $\mathbb{R}^n$ to $\mathbb{R}$

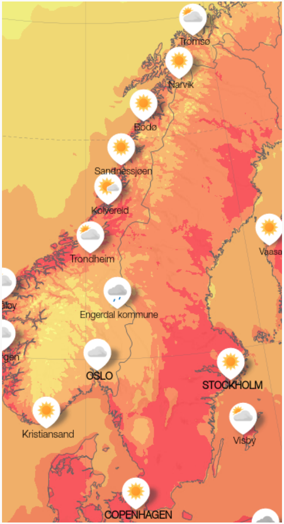
A **function** $f$ of $n$ real variables is a rule that assigns a *unique* real number $f(x_1, x_2, \ldots, x_n)$ to each point $(x_1, x_2, \ldots, x_n)$ in some subset $\mathcal{D}(f)$ of $\mathbb{R}^n$. $\mathcal{D}(f)$ is called the **domain** of $f$. The set of real numbers $f(x_1, x_2, \ldots, x_n)$ obtained from points in the domain is called the **range** of $f$.

As for functions of one variable, the **domain convention** specifies that the domain of a function of $n$ variables is the largest set of points $(x_1, x_2, \ldots, x_n)$ for which $f(x_1, x_2, \ldots, x_n)$ makes sense as a real number, unless that domain is explicitly stated to be a smaller set.

# Example: Temperature

The temperature $T$ in Sweden varies with time and location.
It is a function of three variables, longitude $\lambda$, latitude $\varphi$ and time $t$:

$$T = T(\lambda, \varphi, t)$$



# Example: Waste paper bin

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The area of the bounding surface of a paper basket without a lid depends on the height $h$ and the radius $r$ according to the formula

$$A = \pi r^2 + 2\pi r h$$

where $r > 0$ and $h > 0$.

# Example: Waste paper bin

The area of the bounding surface of a paper basket without a lid depends on the height $h$ and the radius $r$ according to the formula

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where $r > 0$ and $h > 0$.

The area is thus described by the function

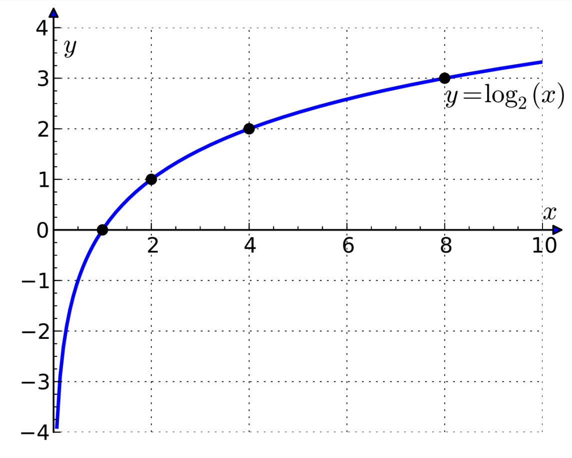
$$A(r, h) = \pi r^2 + 2\pi r h$$

The set $\{(r, h) : r > 0, h > 0\}$ is called the domain of the function.

The range of the function consists of all positive real numbers.

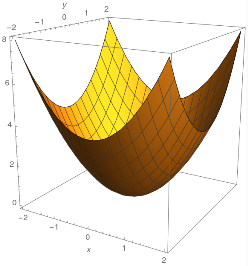
# The graph of a function

The graph of a real-valued function $f$ of one variable is the set of points in the $xy$-plane having coordinates $(x, f(x))$, i.e., a curve.

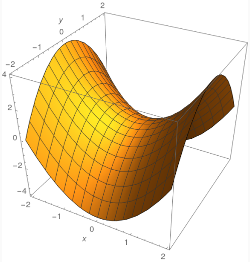


# The graph of a function

The graph of a real-valued function $f$ of two variable is the set of points in the 3-space having coordinates $(x, y, f(x))$, i.e., a surface.




$f(x, y) = x^2 + y^2$




$f(x, y) = x^2 - y^2$

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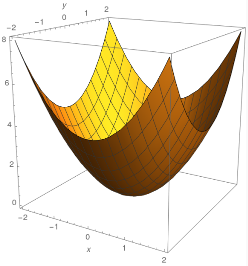
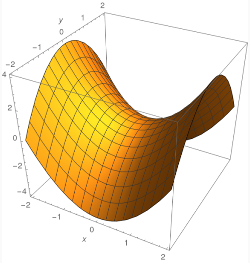



$f(x, y) = x^2 + y^2$

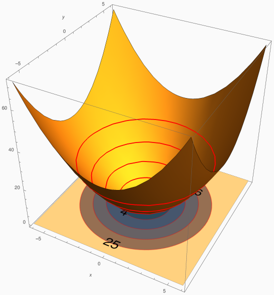



$f(x, y) = x^2 - y^2$

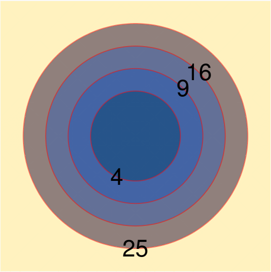
Similarly, a graph of a real-valued function of $n$ variables is an $n$-dimensional surface in $\mathbb{R}^{n+1}$. 😲 We cannot draw graphs for $n \geq 3$ though!

# Level curves

A level curve for a real-value function $f$ of $2$ variables consists of all points $(x, y)$ in the domain of $f$ that satisfy a equation $f(x, y) = C$ for some fixed number $C$.



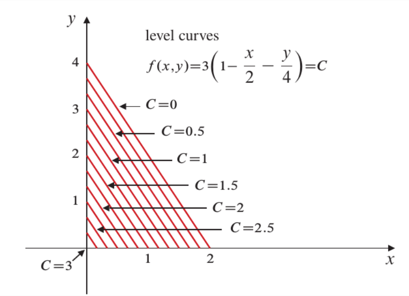

$f = x^2 + y^2$

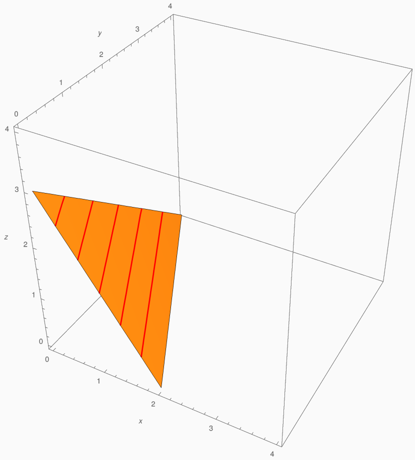



Level curves of $f$


A topographic map

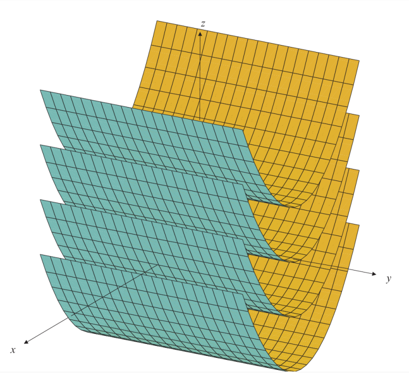
# Example


level curves
$f(x,y)=3\left(1-\frac{x}{2}-\frac{y}{4}\right)=C$
C=0
C=0.5
C=1
C=1.5
C=2
C=2.5
C=3
x
y
1
2
1
2
3
4


x
y
z

# Level surfaces

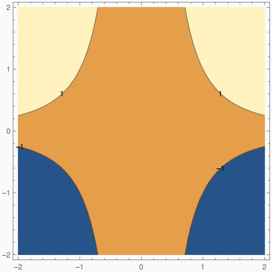
We can also think of a level surfaces of a function of three variables: points $(x, y, z)$ such that $f(x, y, z) = C$ for some fixed number $C$.

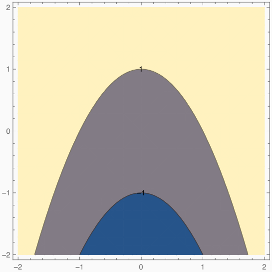


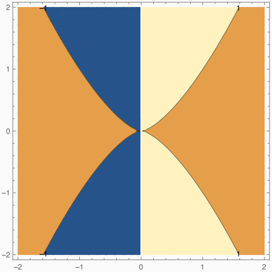

Level surfaces of $f(x, y, z) = x^2 - z$

# Quiz

Let $f(x,y) = x^2 y$. Which of the two picture is the correct level curves $f(x,y) = 1$ and $f(x,y) = -1$?







# Representations of the same curve

Note The same curve in $\mathbb{R}^2$ can be described in several different ways, e.g., level curve, function curve, parameter curve.

Example: The parabola $y = x^2$ can be viewed as

- A level curve for the function $f(x, y) = x^2 - y$.
- The graph of the function $g(x) = x^2$.
- The parametrized curve $(t, t^2)$ or $(t^3, t^6)$, $t \in \mathbb{R}$.

## Representations of the same surface

What about $\mathbb{R}^3$?

Example: The paraboloid $z = x^2 + y^2$ can be viewed as

- A surface area for the function $f(x, y, z) = x^2 + y^2 - z$.
- The graph of the function $z = r^2$ in cylindrical coordinates.
- The parametrized surface $(100t, 2s, (100t)^2 + 4s^2)$, $s, t \in \mathbb{R}$.