# Lecture 04 – 12.2 Limits and Continuity

Several Variable Calculus, 1MA017

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# Summary

Please watch this video before the lecture: 4

Today we will talk about

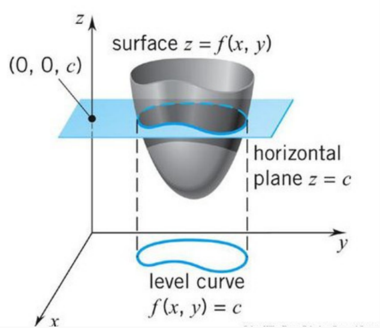
- 12.2 Limits and Continuity

# Review: graphs and level curves

There are two ways to visualize from $\mathbb{R}^2$ to $\mathbb{R}$. We can either draw

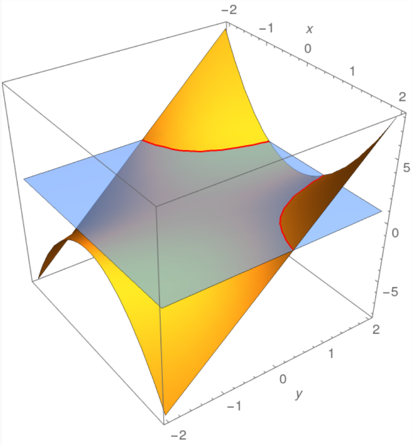
- the function graph $z = f(x, y)$,
- or the level curves $f(x, y) = C$ for some fixed $C$.

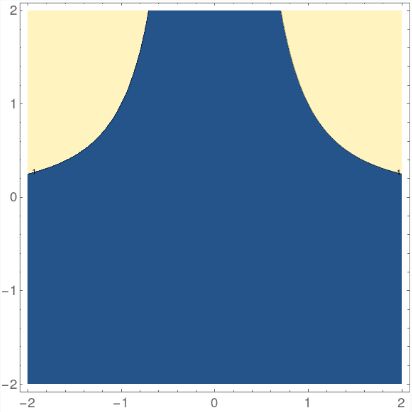
Note: The level curve is also the intersection of the function graph and the plane $z = C$ projected to the $xy$-plane.



# Example

The graph of $f(x,y) = yx^2$ and the level curve $f(x,y) = 1$.



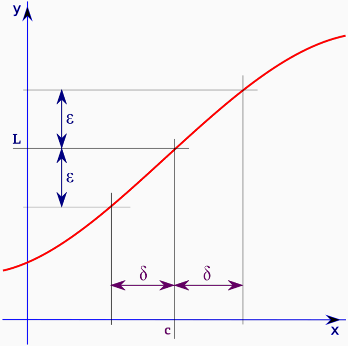
# 12.2 Limits and Continuity

# Review: Limits of functions from $\mathbb{R}$ to $\mathbb{R}$

## Definition

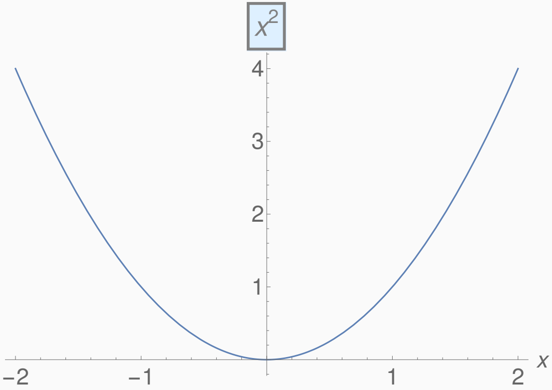
The notation $\lim_{x \to c} f(x) = L$ means that for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - c| < \delta \implies |f(x) - L| < \varepsilon.$$



# Example — Limit of one-variable function

Let $f(x) = x^2$. Then $\lim_{x \to 0} f(x) = 0$.

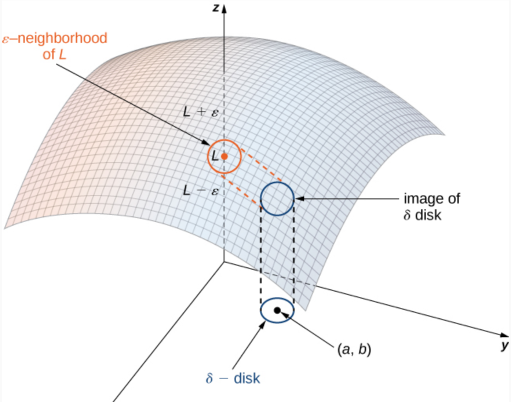


# Limits of functions from $\mathbb{R}^n$ to $\mathbb{R}$

## Definition

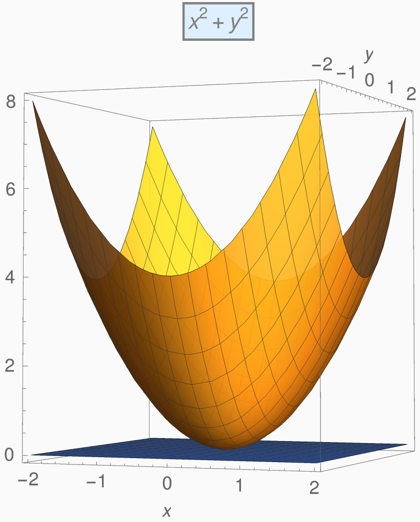
The notation $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L$ means that for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\mathbf{x} - \mathbf{c}| < \delta \implies |f(\mathbf{x}) - L| < \varepsilon$$



# Example — Limit of two-variable function

Let $f(x, y) = x^2 + y^2$. Then $\lim_{(x,y)\to(0,0)} f(x, y) = 0$.



# Functions that are actually $\mathbb{R} \to \mathbb{R}$

In the simplest case when $f(x, y) = g(x)$,

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{x\to a} g(x),$$

if the latter exists.

**Example**

For example, what is

$$\lim_{(x,y)\to(2,1)} 2x = \lim_{x\to 2} 2x = 4.$$

# Rules for computing limits

If $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L$ and $\lim_{\mathbf{x}\to\mathbf{c}} g(\mathbf{x}) = M$ then

- $\lim_{\mathbf{x}\to\mathbf{c}}(f(\mathbf{x}) + g(\mathbf{x})) = L + M$
- $\lim_{\mathbf{x}\to\mathbf{c}}(f(\mathbf{x})g(\mathbf{x})) = LM$
- $\lim_{\mathbf{x}\to\mathbf{c}}(f(\mathbf{x})/g(\mathbf{x})) = L/M$ if $M \neq 0$.

If $H : \mathbb{R} \mapsto \mathbb{R}$ is continuous at $t = L$ then

$$\lim_{\mathbf{x}\to\mathbf{c}} H(f(\mathbf{x})) = H(L)$$

😊 Same as for functions from $f : \mathbb{R} \mapsto \mathbb{R}$!

# Example

(a) $$\lim_{(x,y)\to(2,3)} \left(2x - y^2\right) = 4 - 9 = -5,$$

(b) $$\lim_{(x,y)\to(a,b)} x^2 y = a^2 b,$$

(c) $$\lim_{(x,y)\to(\pi/3,2)} y \sin\left(\frac{x}{y}\right) = 2\sin\left(\frac{\pi}{6}\right) = 1.$$

# Continuity

A real-valued function $f$ of $n$ variables is continuous at the point $\mathbf{a}$ if

$$\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = f(\mathbf{c})$$

In $\mathbb{R}^2$, this is

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

If this is true for all points in the domain of $f$, then $f$ is a continuous function.

# When is a $f : \mathbb{R} \to \mathbb{R}$ discontinuous?

For a one-variable functions, continuity means that the right and left limit values coincide and equal the function value at the point.

There are two reason the function is discontinuous:

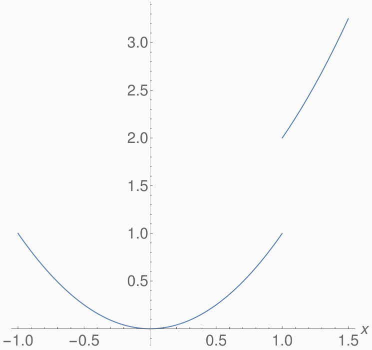
- The left limit does not equal the right limit
- The limit does not equal the function value

# Example

The following function is not continuous at $x = 1$

$$f(x) = \begin{cases} x^2 + 1 & x \geq 1, \\ x^2 & x < 1, \end{cases}$$

because the $\lim_{x \uparrow 1} f(x) \neq \lim_{x \downarrow 1} f(x)$.

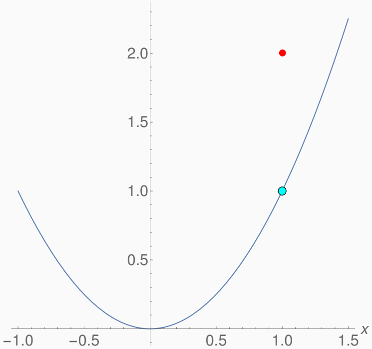


# Example

The following function is not continuous at $x = 1$

$$f(x) = \begin{cases} 2 & x = 1, \\ x^2 & x \neq 1, \end{cases}$$

because the $\lim_{x \to 1} f(x) \neq f(1)$.



# When is a $f : \mathbb{R}^2 \to \mathbb{R}$ discontinuous?

Several-variable functions are more complicated.

There are infinitely many curves that reach the limit point.

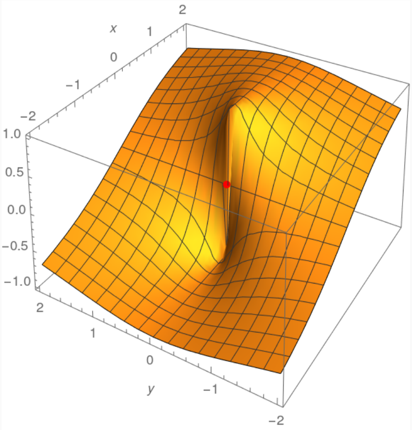
The limit values along all these curves must coincide for the limit of the function to exist.

Only then can we check if the function is continuous

# Example

Is the following function continuous?

$$f(x,y) = \begin{cases} 0 & (x,y) = (0,0), \\ \frac{x}{\sqrt{x^2+y^2}} & \text{otherwise.} \end{cases}$$



# How to think about limit values?

Calculating limit values (when they exist) is generally a lot more difficult than showing that functions don't have limit values (when they don't exist).

One possible tactic:

- First, test along some simple curves, e.g., $x = 0$, $y = 0$, $y = kx$.
- If everything seems to be the same, test polar coordinates.

# Example

Calculate the limit values

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y}{x+y}$$

and

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^4}{\sqrt{4x^2+4y^2}}$$

if they exist.

# Quiz 😲

The function

$$f(x,y) = \begin{cases} 0 & (x,y) = (0,0), \\ \frac{xy}{x^2+y^2} & \text{otherwise.} \end{cases}$$

satisfies

$$\lim_{(x,0)\to(0,0)} f(x,0) = \lim_{(0,y)\to(0,0)} f(0,y) = f(0,0) = 0$$

That is, the limit value along the lines $y = 0$ and $x = 0$ is $0$, which is equal to the function value. Is function continuous? Why?

Hint What is the $\lim_{x\to 0} f(x,x)$?

# Quiz 😲

Does the limit values

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2}$$

and

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + xy + y^2}{x^2 + y^2}$$

exist?

1. Yes, Yes
2. Yes, No
3. No, Yes
4. No, No

# Some fun fact 😊

Continuity is maintained by plus, minus, multiply, division, and composition.

Elementary expressions ($\exp$, $\log$, $\sin$, $\cos$, $\tan$, $\cot$, $\arcsin$, $\arccos$, $\arctan$, etc.) are continuous wherever they are are defined.

**Example**

At what points is the function $f$ given by $f(x, y) = \frac{y}{x}$ continuous?

How to calculate $\lim_{(x,y)\to(1,2)} f(x, y)$ for this function?

# Continuous extension

If $f(x,y)$ is continuous and defined outside $(x,y)=(a,b)$ and if

$$\lim_{(x,y)\to(0,0)} f(x,y) = L,$$

then we can define its continuous extension

$$f_{cont} = \begin{cases} f(x,y) & (x,y) \neq (a,b), \\ L & (x,y) = (a,b). \end{cases}$$

# Example

The function $f(x,y) = \frac{x^2+y^4}{\sqrt{4x^2+4y^2}}$ is not defined at $(0,0)$.

But its continuous extension

$$f_{cont} = \begin{cases} f(x,y) & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

is continuous on the whole $\mathbb{R}^2$.