Lecture 05 – 12.3 Partial Derivatives

Several Variable Calculus, 1MA017

Xing Shi Cai

Autumn 2019

Department of Mathematics, Uppsala University, Sweden

Summary

Please watch this video **before** the lecture: 5

Today we will talk about

- 12.3 Partial Derivatives
- 12.4 Higher-Order Derivatives

Review

Remember:

We have the precise mathematical definitions of limits and continuity.

Functions given by elementary expressions are continuous everywhere they are defined, i.e., if f is such a function and $\mathbf{a}$ is in the domain of f, then

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$$

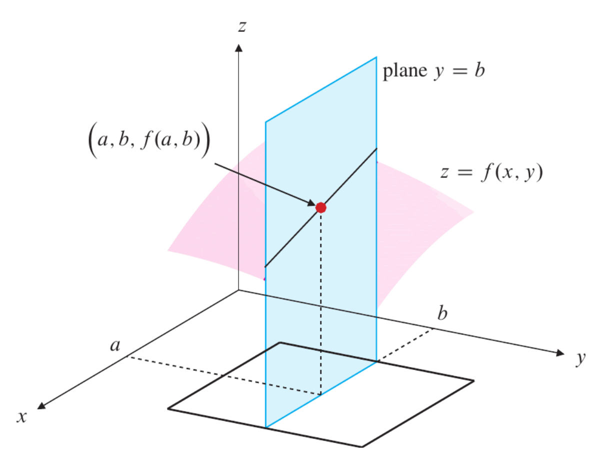
North or east

You have hiked in the mountains for a day and you want to go home as fast possible.
The function $f(x, y)$ gives the altitude of position (x, y).
You can go either east (along the x-axis) or go north (along the y-axis).
How can you tell which direction is better — the altitude degreasers faster?

12.3 Partial Derivatives

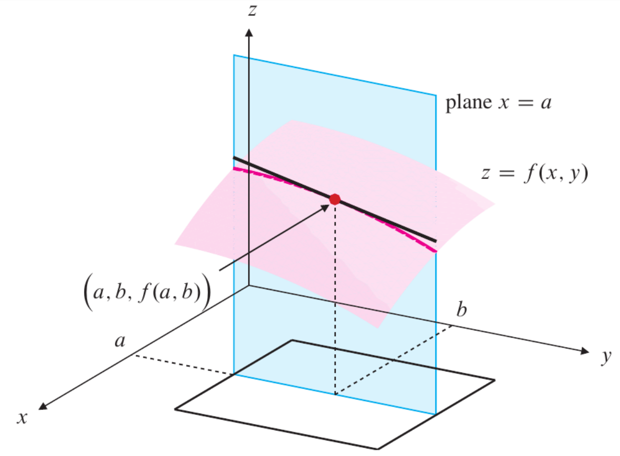
Graphic presentation of partial derivatives

$\frac{\partial f}{\partial x}(a, b)$ is the slope of the intersection curve of the red surface and the blue plane at $x = a$.



Graphic presentation of partial derivatives

$\frac{\partial f}{\partial y}(a, b)$ is the slope of the intersection curve of the red surface and the blue plane at $y = b$.



The definition of partial derivative

Definition

Letting one-variable functions $g(x) = f(x, b)$ and $h(y) = f(a, y)$, the partial derivatives of $f(x, y)$ at (a, b) is defined by

$$f_1(a, b) = \frac{\partial f}{\partial x}(a, b) = \frac{dg}{dx}(a), \qquad f_2(a, b) = \frac{\partial f}{\partial y}(a, b) = \frac{dh}{dy}(b).$$

The definition of partial derivative

Definition

Letting one-variable functions $g(x) = f(x, b)$ and $h(y) = f(a, y)$, the partial derivatives of $f(x, y)$ at (a, b) is defined by

$$f_1(a,b) = \frac{\partial f}{\partial x}(a,b) = \frac{dg}{dx}(a), \qquad f_2(a,b) = \frac{\partial f}{\partial y}(a,b) = \frac{dh}{dy}(b).$$

In other words, you freeze all variables except one and take derive with respect to it.

Notations for partial derivatives

$$f_1,\ f_2,\ \frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ \partial_x f,\ \partial_y f,\ f_x,\ f_y,$$
$$D_x f,\ D_y f,\ D_1 f,\ D_2 f.$$

Example

Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of the functions at $(1, 0)$.

- $f(x, y) = -\left(\frac{x^2}{2} + \frac{y^2}{4}\right)$
- $f(x, y) = x^2 y + y^3$
- $f(x, y) = \sin(xy)$

Quiz

Let $f(x, y) = \sqrt{1 + xy}$, calculate $\frac{\partial f}{\partial y}(2, 1)$.

Answer:

1. $\left\{ \frac{1}{2\sqrt{5}}, \frac{1}{\sqrt{5}} \right\}$
2. $\left\{ \frac{1}{2\sqrt{3}}, \frac{1}{\sqrt{3}} \right\}$
3. $\left\{ \frac{1}{2\sqrt{3}}, \frac{2}{\sqrt{3}} \right\}$
4. $\left\{ \frac{1}{2\sqrt{3}}, \frac{1}{\sqrt{3}} \right\}$

Hint

Quiz

Let $f(x, y) = \arctan \frac{y}{x}$, calculate $\frac{\partial f}{\partial x}(\sqrt{3}, 1)$.

Hint:

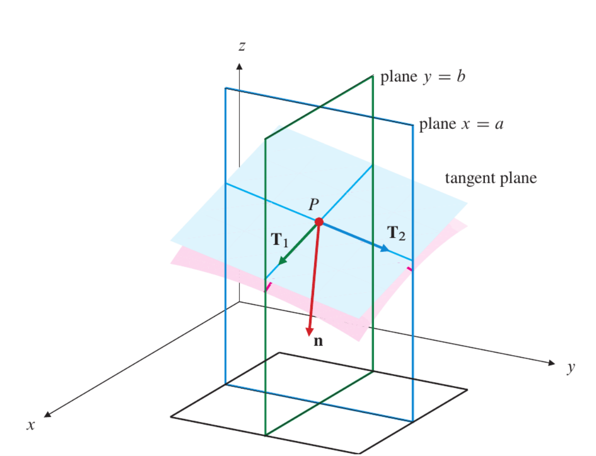
$$\frac{\mathrm{d}}{\mathrm{d}u} \arctan(u) = \frac{1}{1 + u^2}.$$

Answer:

1. $\left\{-\frac{1}{4}, \frac{\sqrt{3}}{4}\right\}$
2. $\left\{-\frac{1}{5}, \frac{\sqrt{3}}{5}\right\}$
3. $\left\{-\frac{1}{4}, \frac{\sqrt{3}}{4}\right\}$
4. $\left\{-\frac{1}{4}, \frac{\sqrt{3}}{4}\right\}$

Tangent plane

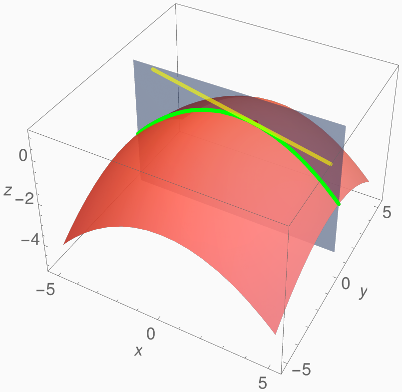
If the graph $z = f(x, y)$ is a "smooth" surface near the point P with coordinates (a, b, f(a,b)), then the graph will have a tangent plane and a normal line at P.



Example: Find the tangent plane

To find the tangent plane of $z = f(x, y)$ at the point $P_0 = (a, b, f(a, b))$, we

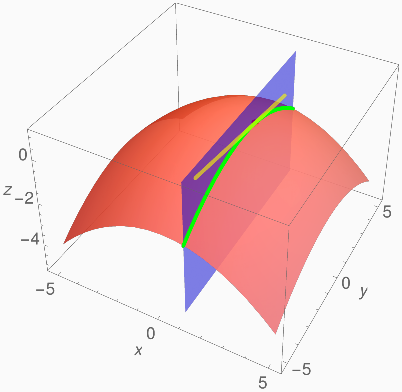
1. Find the intersection curve of the surface with the plane $y = b$. Determine T_1, the tangent vector for this curve at the point P_0.



Example: Find the tangent plane

To find the tangent plane of $z = f(x, y)$ at the point $P_0 = (a, b, f(a, b))$, we

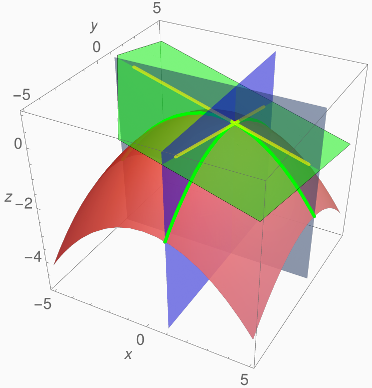
2. Find the intersection curve of the surface with the plane $x = a$. Determine T_2, the tangent vector for this curve at the point P_0.



Example: Find the tangent plane

To find the tangent plane of $z = f(x, y)$ at the point $P_0 = (a, b, f(a, b))$, we

3. Find the plane parallel to both T_1 and T_2 which goes through P_0. This is the tangent plane.

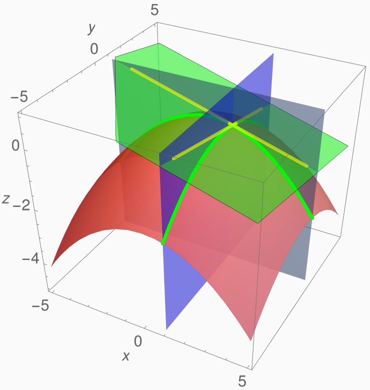


Formula for the tangent plane

With a bit help from linear algebra, we find the tangent plane of $z = f(x, y)$ at $(a, b, f(a, b))$ to be

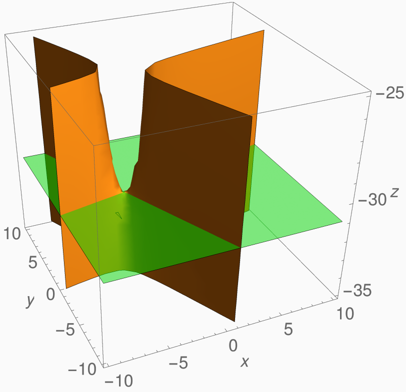
$$z = L(x, y) = f(a, b) + f_1(a, b)(x - a) + f_2(a, b)(y - b)$$

This is also called the linearisation of f at (a, b).



Example — tangent plane

Determine if the area $z = x^2 - 4xy - 2y^2 + 12x - 12y - 1$ has any tangent plane that is horizontal (i.e., parallel to the xy-plane). Then find the equation for this tangent plane and the point where it touches the surface.



Example — tangent plane

Find the tangent plane at the point $(1, 2, 3)$ of the surface $z = f(x, y)$ where

$$f(x, y) = \frac{x^2 y}{y - 1} + 1$$

and use it to approximate the function value $f(1.2, 2.3)$

Example — tangent plane

Find the tangent plane at the point $(1, 2, 3)$ of the surface $z = f(x, y)$ where

$$f(x, y) = \frac{x^2 y}{y - 1} + 1$$

and use it to approximate the function value $f(1.2, 2.3)$

The partial derivatives are

$$f_1(x, y) = \frac{2xy}{y - 1}, \qquad f_2(x, y) = \frac{x^2}{y - 1} - \frac{x^2 y}{(y - 1)^2}.$$

So the linearisation is

$$L(x, y) = f(1, 2) + f_1(1, 2)(x - 1) + f_2(1, 2)(y - 2) = 1 + 4x - y.$$

The approximation $L(1.2, 2.3) = 3.5$, while $f(1.2, 2.3) = 3.54769$.

Quiz: tangent plane

The tangent plane of $z = f(x, y)$ at $(a, b, f(a, b))$ is given by

$$z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial x}(a, b)(y - b)$$

Find the tangent plane of $f(x, y) = x^2y + y^3 - 7$ at $(1, 2, 3)$.

Which one is the correct answer?

1. $2 + 3(x - 2) + 13(y - 2)$
2. $3 + 4(x - 1) + 13(y - 2)$
3. $3 + 4(x - 1) + 13(y - 3)$
4. $3 + 2(x - 2) + 13(y - 3)$

Higher-Order Derivatives

Higher-order derivatives

The partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of the function $f(x, y)$ are themselves two-variable functions.

So we can also take their partial derivatives and there are 4 possible ways to do this

$$\frac{\partial^2 f}{\partial x^2} = f_{11}(x, y), \qquad \frac{\partial^2 f}{\partial y \partial x} = f_{12}(x, y),$$
$$\frac{\partial^2 f}{\partial x \partial y} = f_{21}(x, y), \qquad \frac{\partial^2 f}{\partial y^2} = f_{22}(x, y).$$

These are called second order partial derivatives.

Higher-order derivatives

The partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of the function $f(x, y)$ are themselves two-variable functions.

So we can also take their partial derivatives and there are 4 possible ways to do this

$$\frac{\partial^2 f}{\partial x^2} = f_{11}(x, y), \qquad \frac{\partial^2 f}{\partial y \partial x} = f_{12}(x, y),$$
$$\frac{\partial^2 f}{\partial x \partial y} = f_{21}(x, y), \qquad \frac{\partial^2 f}{\partial y^2} = f_{22}(x, y).$$

These are called second order partial derivatives.

😕 If f has 3 variables (x, y, z), how many possible second-order partial derivatives can f have?

Example

Compute the four partial second order derivatives to $f(x, y) = x^2y + y^3 - 7$. Do you notice something strange?

Example

Compute the four partial second order derivatives to $f(x, y) = x^2y + y^3 - 7$. Do you notice something strange?

If the mixed partial second derivatives are continuous they are equal, i.e.,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

Equality of mixed partials

Theorem 1, chap. 12.4

Suppose that two mixed nth-order partial derivatives of a function f involve the same differentiations but in different orders. If those partials are continuous at a point P; and if f and all partials of f of order less than n are continuous in a neighbourhood of P; then the two mixed partials are equal at the point P.

The Laplace equation

A function $f(x, y)$ satisfies the Laplace equation at (a, b) if

$$\nabla^2(f(x, y)) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Remark

A function of two variables is said to be harmonic there if it satisfies Laplace's equation. Such functions and are used to model various physical quantities such as electric and magnetic potential fields.

For example, $\log(x^2 + y^2)$ satisfies the Laplace equation on the whole xy-plane except at $(0, 0)$.

Pierre-Simon Laplace

French mathematician. 23 March 1749 –5 March 1827

Quiz — Laplace equation

Do these two functions satisfy the Laplace equations on the whole xy-plane?

$$f(x,y) = e^{3x}\cos(5y),$$
$$h(x,y) = e^{3x}\cos(3y).$$

Answer

1. Yes, yes
2. Yes, no
3. No, yes
4. No, no

Wave equation

The function

$$w(x,t) = f(x - ct) + g(x + ct)$$

satiesfies the wave equation, i.e.,

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$