Lecture 06 – 12.6 Linear Approximations, Differentiability, and Differentials

Several Variable Calculus, 1MA017

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Summary

Please watch this video before the lecture: 6

Today we will talk about

- 12.6 Linear Approximations, Differentiability, and Differentials

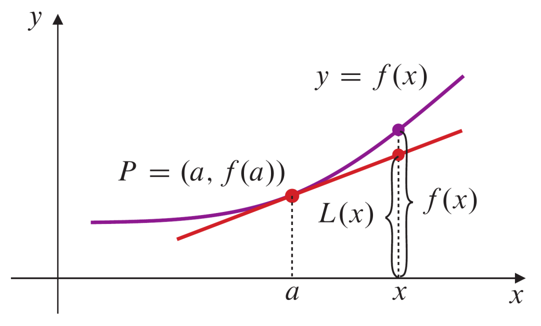
Linearisation

Linearisation for functions from $\mathbb{R}$ to $\mathbb{R}$

The tangent line of the graph $f(x)$ at $x = a$ gives an approximation of $f(x)$ for x near a, i.e.,

$$f(x) \approx L(x) = f(a) + f'(a)(x - a).$$

The function $L(x)$ is called the linearisation of f at a.



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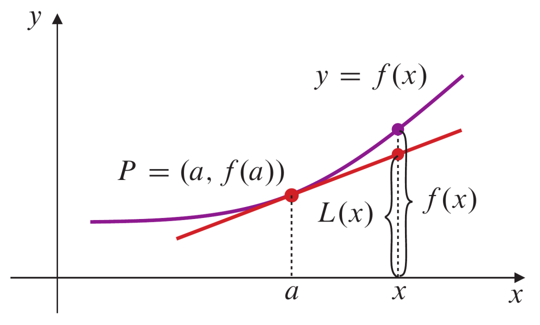
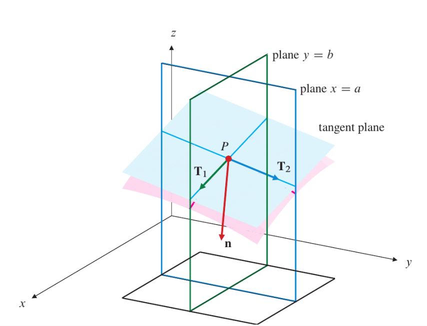


The tangent plane of the graph $f(x, y)$ at (a, b) also gives an approximation of $f(x, y)$ for x near a, i.e.,

$$f(x, y) \approx L(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - a).$$

The function $L(x, y)$ is called the linearisation of f at (a, b).



Example — Linearisation

Find the linearisation of

$$f(x, y) = \frac{x^2 y}{y - 1} + 1$$

at $1, 2$ and use it to approximate the function value $f(1.2, 2.3)$

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The partial derivatives are

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So the linearisation is

$$L(x, y) = f(1, 2) + \frac{\partial f}{\partial x}(1, 2)(x - 1) + \frac{\partial f}{\partial y}(1, 2)(y - 2) = 1 + 4x - y.$$

The approximation $L(1.2, 2.3) = 3.5$, while $f(1.2, 2.3) = 3.548$.

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Find the linearisation of

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at $(2,0)$ and use it to approximate the function value $f(2.2,-0.2)$

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So the linearisation is

$$L(x,y) = f(1,2) + \frac{\partial f}{\partial x}(1,2)(x-2) + \frac{\partial f}{\partial y}(1,2)(y-0) = \frac{4x}{3} + \frac{y}{3} + \frac{1}{3}.$$

The approximation $L(2.2,-0.2) = 3.2$, while $f(2.2,-0.2) = 3.217$.

Quiz: Linearisation

Which of the following is $L(x, y)$, the linearisation of $f(x, y) = e^{x^2 y - 1}$ at $(-1, 1)$? (Hint)

1. $-2 - 2x + y$
2. $-2 - 2x + 2y$
3. $-3 - x + 2y$
4. $-3 - 2x + 2y$

What is $L(-0.9, 1.1)$?

1. 0.8
2. 0.9
3. 1.0
4. 1.1

Differentiability

Good approximation for $f : \mathbb{R} \to \mathbb{R}$

If $f'(a)$ exists, then

- $f(x)$ is continuous at $x = a$,
- The approximation

$$f(x) \approx L(x) = f(a) + f'(a)(x - a),$$

  is "good", i.e.,

$$\lim_{h \to 0} \frac{f(a+h) - L(a+h)}{h} = 0$$

Good approximation for $f : \mathbb{R}^2 \to \mathbb{R}$??

Question: Is it also true that if the partial derivatives

$$\frac{\partial f}{\partial x}(a, b), \qquad \frac{\partial f}{\partial y}(a, b),$$

exist, then

- $f(x, y)$ is continuous at (a, b),
- The approximation

  $$f(x, y) \approx L(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - a),$$

  is "good".

Good approximation for $f : \mathbb{R}^2 \to \mathbb{R}$??

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  $$f(x,y) \approx L(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-a),$$

  is "good".

Answer: NO 😲

What does "good" mean?

A function $f(x, y)$ is differentiable at (a, b) if

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h, b+k) - L(a+h, b+k)}{\sqrt{h^2+k^2}} = 0$$

where

$$L(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x-a) + \frac{\partial f}{\partial y}(a, b)(y-a),$$

In other words, a function is differentiable when its linearisation is a good approximation.

Properties of differentiable functions

If $f(x,y)$ is differentiable at (a,b) then

- $f(x,y)$ is continuous at (a,b)
- Partial derivatives $\frac{\partial f}{\partial x}(a,b)$ and $\frac{\partial f}{\partial y}(a,b)$ must exist.

The inverse is not true! 😮

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The inverse is not true! 😲

But if partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous in a neighbourhood of (a, b), then f is differentiable at (a, b). (See Chapter 12.6 Theorem 3 and 4)

Example: Partial derivatives does not imply continuity

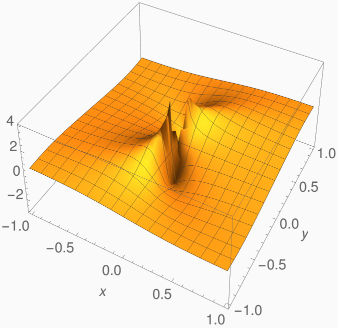
There are functions with partial derivatives but are not continuous. Consider

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^4}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

We have

$$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0.$$

But the function is not continuous at $(0,0)$! 😲

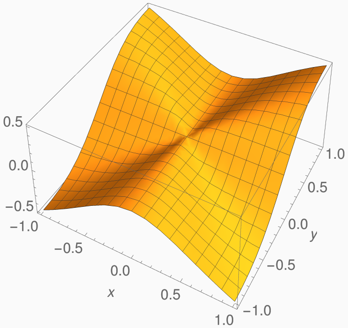


Example: continuous but not differentiable

The function

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0), \end{cases}$$

is continuous.



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is continuous.

But its partial derivatives

$$f_1(0,0) = 0, \quad f_2(0,0) = 0.$$

Thus it's linearisation is

$$L(x,y) = f(0,0) + 0(x-0) + 0(y-0) = f(0,0) = 0.$$

Using polar coordinates, we have

$$\frac{|f(x,y) - L(x,y)|}{\sqrt{x^2+y^2}} = \cos(\theta)^2 \sin(\theta).$$

Thus the function is not differentiable.

Proof: Differentiability implies continuity

A function $f(x, y)$ is differentiable at (a, b) if

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h, b+k) - (f(a,b) + f_1(a,b)h + f_2(a,b)k)}{\sqrt{h^2+k^2}} = 0,$$

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which implies

$$\lim_{(h,k)\to(0,0)} f(a+h,b+k)-(f(a,b)+f_1(a,b)h+f_2(a,b)k) = 0.$$

Proof: Differentiability implies continuity

A function $f(x, y)$ is differentiable at (a, b) if

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which implies

$$\lim_{(h,k)\to(0,0)} f(a+h, b+k) - (f(a,b) + f_1(a,b)h + f_2(a,b)k) = 0.$$

Therefore

$$\lim_{(h,k)\to(0,0)} f(a+h, b+k) - f(a,b) = 0.$$

Thus $f(a, b)$ is continuous at (a, b).

Functions from $\mathbb{R}^n$ to $\mathbb{R}^m$

The Jacobian matrix

For vector-valued function of multi-variables

$$\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_m(\mathbf{x})) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)),$$

the matrix $m \times n$ matrix

$$\mathbf{f}'(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

is called the Jacobian matrix of $\mathbf{f}$.

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is called the Jacobian matrix of $\mathbf{f}$.

For the function $f(x, y)$ from $\mathbb{R}^2$ to $\mathbb{R}$, this is

$$f'(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} f_1(x, y) & f_2(x, y) \end{bmatrix}$$

Linearisation

The Linearisation for $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ is defined by

$$\mathbf{L}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + \mathbf{f}'(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

For the function $f(x, y)$, this is

$$\begin{aligned} L(x,y) &= f(a,b) + \begin{bmatrix} f_1(x,y) & f_2(x,y) \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} \right) \\ &= f(a,b) + \begin{bmatrix} f_1(x,y) & f_2(x,y) \end{bmatrix} \begin{bmatrix} x - a \\ y - b \end{bmatrix} \\ &= f(a,b) + f_1(a,b)(x-a) + f_2(a,b)(y-b). \end{aligned}$$

Differentiability

The function $\mathbf{f}(\mathbf{x})$ is differentiable at a point $\mathbf{a}$, if

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - \mathbf{f}'(\mathbf{a})(\mathbf{x}-\mathbf{a})|}{|\mathbf{x}-\mathbf{a}|}$$

i.e., $\mathbf{f}$ is well approximated by its linearisation

$$\mathbf{f}(\mathbf{x}) \approx \mathbf{L}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + \mathbf{f}'(\mathbf{a})(\mathbf{x}-\mathbf{a}).$$

A sufficient condition for $\mathbf{f}(\mathbf{x})$ to be differential is all the partial derivatives in $\mathbf{f}'(\mathbf{x})$ are continuous.

Example — Jacobian matrices

Compute the Jacobian matrices and linearisation at of

- $\mathbf{f}(x) = (x^2, 3x^2, x)$ at a.
- $f(x, y, z) = (x^2 + z - y^3)$ at (a, b, c).
- $\mathbf{f}(x, y) = (x^2 - y, y^2 + \cos(x))$ at (a, b).

Quiz: Linearisation

Compute the Jacobian matrices and linearisations of

$$\mathbf{f}(x, y) = (x^2 \ln(y), xy)$$

at $(2, 1)$

1. $(-4 + 2x + 4y, -1 + x + 2y)$
2. $(-4 + x + 3y, -2 + x + y)$
3. $(-4 + 4y, -2 + x + 2y)$
4. $(-4 + 3y, -2 + 2y)$