Lecture 07 – 12.5 The Chain Rule

Several Variable Calculus, 1MA017

Xing Shi Cai

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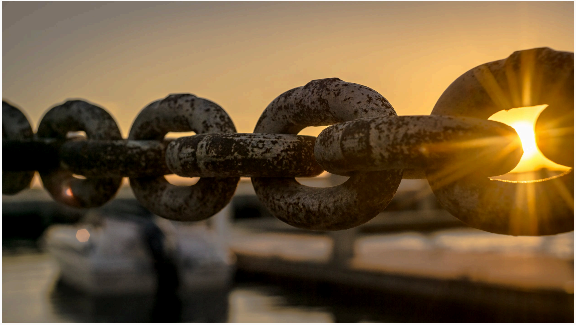
Department of Mathematics, Uppsala University, Sweden

Summary

Please watch this video before the lecture: 7

Today we will talk about

- 12.5 The Chain Rule



The hiking question

- You went for a hike last weekend.
- The function $f(x, y)$ gives the altitude of position (x, y).
- Your position at time t is given by $x = u(t)$ and $y = v(t)$.
- How fast is your altitude changes with respect to t?

Review: The chain rule for $\mathbb{R} \mapsto \mathbb{R} \mapsto \mathbb{R}$

For real-valued one-variable functions $h(x) = g \circ f(x) = g(f(x))$ we have

$$h'(x) = g'(f(x))f'(x).$$

Example

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin\big(x^2 + 2x\big) = \cos\big(x^2 + 2x\big)(2x + 2).$$

The chain rule for $\mathbb{R} \mapsto \mathbb{R}^2 \mapsto \mathbb{R}$

Let $g(t) = f(u(t), v(t))$. Then

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{\partial f}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial f}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}t},$$

if

- the partial derivatives $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ are continuous,
- and the derivatives $\frac{\mathrm{d}u}{\mathrm{d}t}$ and $\frac{\mathrm{d}v}{\mathrm{d}t}$ exist.

The chain rule for $\mathbb{R} \mapsto \mathbb{R}^2 \mapsto \mathbb{R}$

To prove

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{\partial f}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial f}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}t},$$

note that

$$\begin{aligned} g'(t) &= \lim_{h\to 0} \frac{g(t+h)-g(t)}{h} = \lim_{h\to 0} \frac{f\big(u(t+h), v(t+h)\big) - f\big(u(t), v(t)\big)}{h} \\ &= \lim_{h\to 0} \frac{f\big(u(t+h), v(t+h)\big) - f\big(u(t), v(t+h)\big)}{h} \\ &\qquad + \lim_{h\to 0} \frac{f\big(u(t), v(t+h)\big) - f\big(u(t), v(t)\big)}{h}. \end{aligned}$$

Example

Let $f(x,y) = \sin\big(x^2 y\big)$, where $x = t^3$ and $y = t^2 + t$. Find

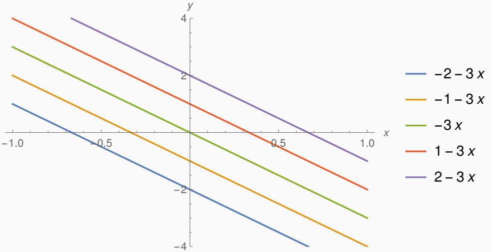
$$\frac{\mathrm{d}}{\mathrm{d}t} f(x(t), y(t)).$$

Exam question

Suppose that $f(x, y)$ satisfies the differential equation

$$\frac{\partial f}{\partial x} = 3\frac{\partial f}{\partial y}$$

in the whole plane. Show that $f(x, y)$ is a constant on every line that is parallel to the line $3x + y = 1$.



Solution

Quiz

Consider the function

$$f(x, y) = x^2/a^2 + y^2/b^2,$$

and $g(t) = f(\cos(t), \sin(t))$. Which of the following is $\frac{\mathrm{d}g}{\mathrm{d}t}$

1. $\dfrac{\sin(t)\cos(t)}{b^2} - \dfrac{2\sin(t)\cos(t)}{a^2}$
2. $\dfrac{2\sin(t)\cos(t)}{b^2} - \dfrac{2\sin(t)\cos(t)}{a^2}$
3. $\dfrac{2\sin(t)\cos(t)}{b} - \dfrac{\sin(t)\cos(t)}{a}$
4. $\dfrac{\sin(t)\cos(t)}{a^2} - \dfrac{2\sin(t)\cos(t)}{b^2}$

The chain rule for $\mathbb{R}^2 \mapsto \mathbb{R}^2 \mapsto \mathbb{R}$

Let $g(s,t) = f(\mathbf{x}(s,t)) = f(u(s,t), v(s,t))$. If

- the partial derivatives $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ are continuous,
- and the partial derivatives $\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}, \frac{\partial v}{\partial s}, \frac{\partial v}{\partial t}$ exist,

then

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial s}, \qquad \frac{\partial g}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial t}.$$

Idea — Treat t (s) as constant, apply chain rule to s (t).

The chain rule for $\mathbb{R}^2 \mapsto \mathbb{R}^2 \mapsto \mathbb{R}$

Let $g(s,t) = f(\mathbf{x}(s,t)) = f(u(s,t), v(s,t))$. If

- the partial derivatives $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ are continuous,
- and the partial derivatives $\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}, \frac{\partial v}{\partial s}, \frac{\partial v}{\partial t}$ exist,

then

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial s}, \qquad \frac{\partial g}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial t}.$$

Idea — Treat t (s) as constant, apply chain rule to s (t).

Note — Another way to write the rules is

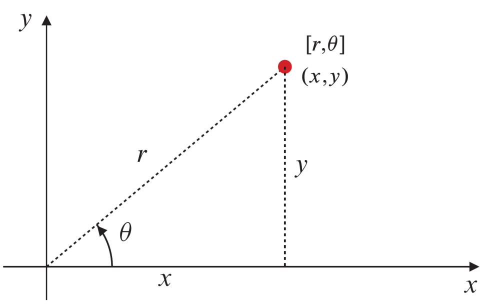
$$g'(s,t) = \begin{pmatrix} \frac{\partial g}{\partial s} & \frac{\partial g}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{pmatrix} = f'(\mathbf{x})\mathbf{x}'.$$

Example — Partial derivatives in the polar coordinates

Let $f(x, y) = xy$. Compute

$$\frac{\partial f(x(r,\theta), y(r,\theta))}{\partial r}, \qquad \frac{\partial f(x(r,\theta), y(r,\theta))}{\partial \theta},$$

with $x = r\cos(\theta)$ and $y = r\sin(\theta)$.



Exam problem

The functions $f(x, y)$ and $g(r, \theta)$ satisfies the equation

$$g(r, \theta) = f(r\cos\theta, r\sin\theta).$$

Assume that

$$\frac{\partial g}{\partial r}\left(2, \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}, \qquad \frac{\partial g}{\partial \theta}\left(2, \frac{\pi}{3}\right) = 9.$$

Compute

$$\frac{\partial f}{\partial x}\left(1, \sqrt{3}\right), \qquad \frac{\partial f}{\partial y}\left(1, \sqrt{3}\right).$$

The chain rule for $\mathbb{R}^n \mapsto \mathbb{R}^m \mapsto \mathbb{R}^k$

Let $\mathbf{g}(\mathbf{x})$ be a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. Let $\mathbf{f}(\mathbf{y})$ be a function from $\mathbb{R}^m$ to $\mathbb{R}^k$. Let $\mathbf{h}(\mathbf{x}) = \mathbf{f}(\mathbf{g}(\mathbf{x}))$. Then

$$\mathbf{h}'(\mathbf{x}) = \mathbf{f}'(\mathbf{g}(\mathbf{x}))\mathbf{g}'(\mathbf{x}).$$

In the case $n = 1$, $m = 2$, $k = 1$, this is just

$$\begin{aligned}\mathbf{h}'(x_1) = \mathbf{f}'(y_1, y_2)\mathbf{g}'(x_1) &= \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \end{bmatrix} \begin{bmatrix} \frac{\partial g_1}{\partial x_1} \\ \frac{\partial g_2}{\partial x_2} \end{bmatrix} \\ &= \frac{\partial f_1}{\partial y_1}\frac{\partial g_1}{\partial x_1} + \frac{\partial f_1}{\partial y_2}\frac{\partial g_2}{\partial x_2}\end{aligned}$$