Lecture 08 – 12.7 Gradients and Directional Derivatives

Several Variable Calculus, 1MA017

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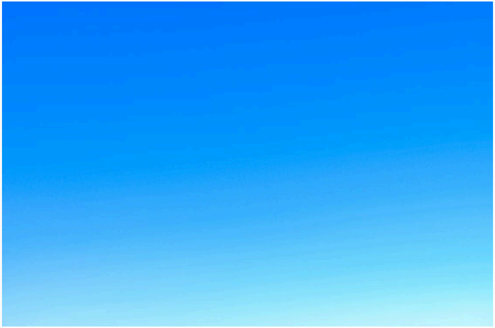
Summary

Please watch this video before the lecture: 8

Today we will talk about

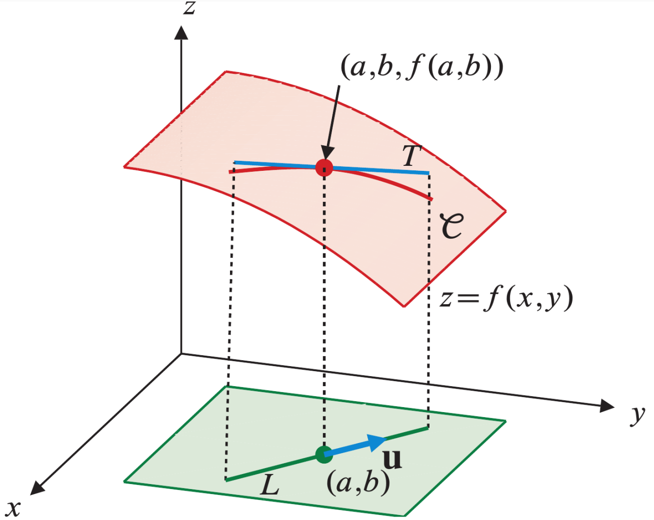
- 12.7 Gradients and Directional Derivatives



Another hiking question

- You are climbing a mountain.
- The function $f(x, y)$ gives the altitude of position (x, y).
- Your current position is (a, b).
- Which direction should you go to increase your altitude as fast/slow as possible?

The directional derivative



The directional derivative

Let $\mathbf{u}$ be a n-dimension unit vector.

Let $h(t) = f(\mathbf{a} + t\mathbf{u})$.

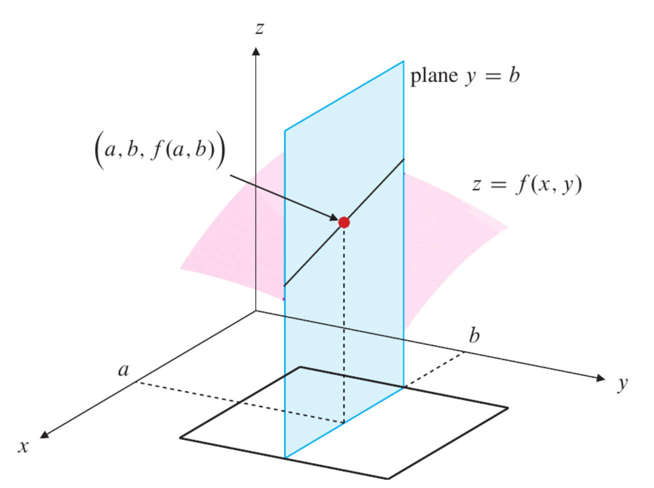
The directional derivative of $f : \mathbb{R}^n \to \mathbb{R}$ at a point $\mathbf{a}$ is defined by

$$D_{\mathbf{u}} f(\mathbf{a}) = h'(0).$$

Interpretation: $D_{\mathbf{u}} f(\mathbf{a})$ indicates the speed of change of f at the point $\mathbf{a}$ in the direction of $\mathbf{u}$.

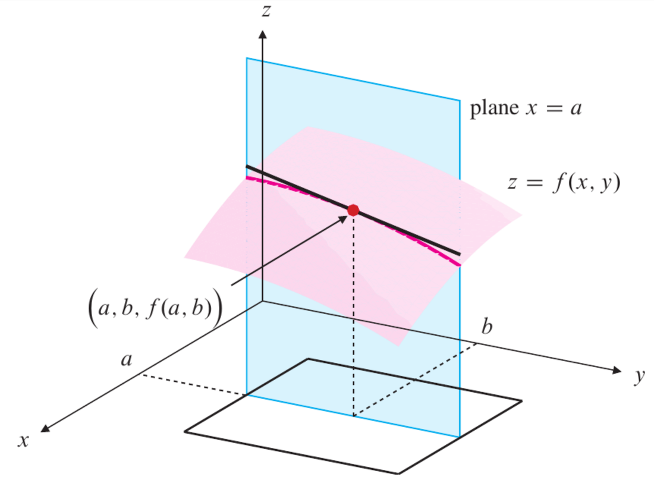
Partial derivatives are also directional derivatives

The partial derivative $\frac{\partial f}{\partial x} = D_{\mathbf{u}} f$ for $\mathbf{u} = (1, 0)$, i.e., when $\mathbf{u}$ is the unit vector along the direction of x axis.



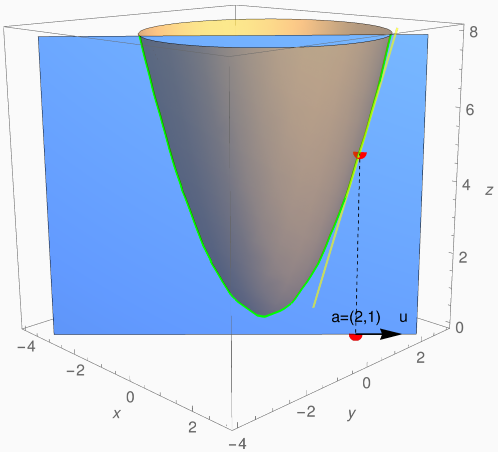
Partial derivatives are also directional derivatives

The partial derivative $\frac{\partial f}{\partial y} = D_{\mathbf{u}} f$ for $\mathbf{u} = (0, 1)$, i.e., when $\mathbf{u}$ is the unit vector along the direction of y axis.



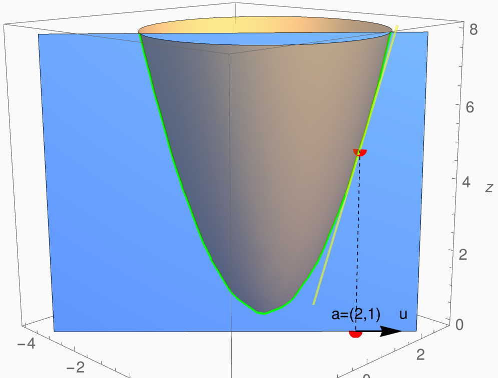
Example

Compute the directional derivative of f at $\mathbf{a} = (2, 1)$ on the direction $\mathbf{u} = (\cos(\pi/4), \sin(\pi/4))$ if $f(x, y) = x^2 + y^2$.



Example – Interpretation

- The point $\mathbf{a}$ and the vector $\mathbf{u}$ decide the blue plane.
- The function $h(t) = f(\mathbf{a} + t\mathbf{u})$ gives the green curve.
- The yellow line is the tangent line of $h(t)$ at $t = 0$.
- The result $D_{\mathbf{u}} f(\mathbf{a}) = h'(0) = 3\sqrt{2}$ is the slope of the yellow line.



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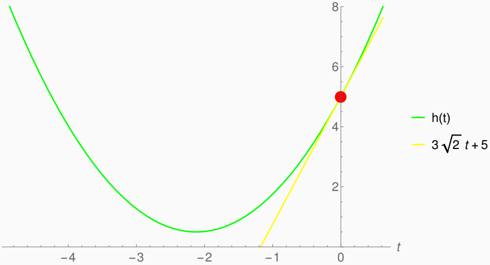


The gradient of a function from $\mathbb{R}^2$ to $\mathbb{R}$

The gradient of a function $f(x, y)$ is the vector

$$\nabla f(x, y) = \mathbf{grad} f(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right).$$

Example

Compute $\nabla f(x, y)$ and $\nabla f(1, 2)$ for $f(x, y) = x^2 y$.

The gradient of a function from $\mathbb{R}^n$ to $\mathbb{R}$

The gradient of a function $f(x_1, \dots, x_n)$ is the vector

$$\nabla f(x_1, \dots, x_n) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right).$$

Example

Compute $\nabla f(x, y, z)$ and $\nabla f(1, -1, 2)$ for

$$f(x, y, z) = x^2 y + e^{yz+2}.$$

Gradients and directional derivatives

If $f(x_1, \dots, x_n)$ is differentiable at $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{u} = (u_1, \dots, u_n)$ is a unit vector, then

$$D_{\mathbf{u}} f(\mathbf{a}) = \mathbf{u} \cdot \nabla f(\mathbf{a}).$$

To see this, let $\mathbf{x}(t) = \mathbf{a} + \mathbf{u}t$ and note that

$$\begin{aligned} D_{\mathbf{u}} f(\mathbf{a}) &= \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t)) \Big|_{t=0} = f'(\mathbf{x}(0))\mathbf{x}'(0) \\ &= \begin{pmatrix} f_1(\mathbf{a}) & f_2(\mathbf{a}) & \dots & f_n(\mathbf{a}) \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \\ &= f_1(\mathbf{a})u_1 + f_2(\mathbf{a})u_2 + \dots + f_n(\mathbf{a})u_n = \mathbf{u} \cdot \nabla f(\mathbf{a}), \end{aligned}$$

where we apply the chain rule for $f(\mathbf{x}(t))$.

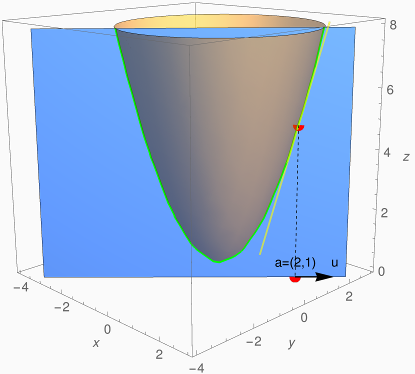
Example

Use the formula

$$D_{\mathbf{u}}f(\mathbf{a}) = \mathbf{u} \cdot \nabla f(\mathbf{a}).$$

to compute the directional derivative of f at $(2, 1)$ on the direction $(\mathbf{u} = \cos(\pi/4), \sin(\pi/4))$ for $f(x, y) = x^2 + y^2$.

The result should be the same as before.



Quiz

Use the formula

$$D_{\mathbf{u}}f(\mathbf{a}) = \mathbf{u} \cdot \nabla f(\mathbf{a}).$$

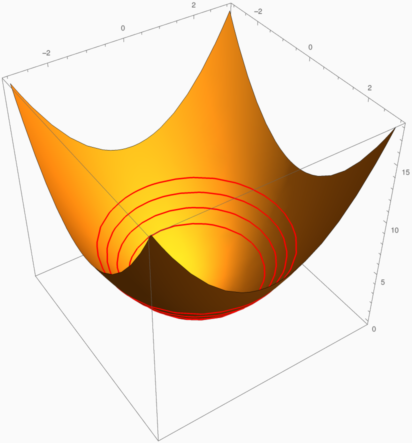
to find the directional derivative for $f(x, y) = x^2 y$ at the point $(2, 3)$ on the direction $\mathbf{u} = (\frac{3}{5}, \frac{4}{5})$.

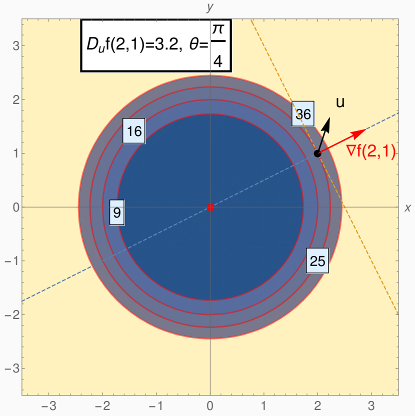
Which of the following is the correct answer?

1. $\frac{51}{5}$
2. $\frac{52}{5}$
3. $\frac{53}{5}$
4. $\frac{54}{5}$

Example

Calculate $Df_{\mathbf{u}}(\mathbf{a})$ at point $\mathbf{a} = (2, 1)$ in the direction $\mathbf{u} = (\cos\theta, \sin\theta))$ if $f(x, y) = x^2 + y^2$.





The geometric properties of the gradient

Thus if f is differentiable at a point $\mathbf{a}$, then at the point $\mathbf{a}$

- If $\nabla f(\mathbf{a}) \neq 0$, then $\nabla f(\mathbf{a})$ is orthogonal to the level curve passing through $\mathbf{a}$.

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- The directional derivatives along the level curves are zero.

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- If $\nabla f(\mathbf{a}) \neq 0$, then $\nabla f(\mathbf{a})$ is orthogonal to the level curve passing through $\mathbf{a}$.
- The directional derivatives along the level curves are zero.
- The function increase fastest along the direction $\nabla f(\mathbf{a})$.

The geometric properties of the gradient

Thus if f is differentiable at a point $\mathbf{a}$, then at the point $\mathbf{a}$

- If $\nabla f(\mathbf{a}) \neq 0$, then $\nabla f(\mathbf{a})$ is orthogonal to the level curve passing through $\mathbf{a}$.
- The directional derivatives along the level curves are zero.
- The function increase fastest along the direction $\nabla f(\mathbf{a})$.
- The function decreases fastest along the direction $-\nabla f(\mathbf{a})$.

The geometric properties of the gradient

Suppose the level curve has a parametrization $\mathbf{r}(t) = (x(t), y(t))$. Then by the chain rule

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{r}(t)) &= f'(\mathbf{r}(t))\mathbf{r}'(t) = \begin{pmatrix} f_1 & f_2 \end{pmatrix} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} \\
&= f_1 x'(t) + f_2 y'(t) \\
&= (f_1, f_2) \cdot (x'(t), y'(t)) \\
&= \nabla f(x, y) \cdot (x'(t), y'(t))
\end{aligned}$$

Since the function does not change its value along a level curve, this derivative must be 0.

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\end{aligned}$$

Since the function does not change its value along a level curve, this derivative must be 0.

So $\nabla f(x, y)$ and $(x'(t), y'(t))$ must be orthogonal!

The geometric properties of the gradient

The directional derivative is given by

$$D_{\mathbf{u}}f(\mathbf{a}) = \mathbf{u} \cdot \nabla f(\mathbf{a}) = |\mathbf{u}||\nabla f(\mathbf{a})| \cos\theta = |\nabla f(\mathbf{a})| \cos\theta,$$

where θ is the angel between $\mathbf{u}$ and $\nabla f(\mathbf{a})$.

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So this is maximized when $\theta = 0$, i.e., when $\mathbf{u}$ and $\nabla f(\mathbf{a})$ are of the same direction.

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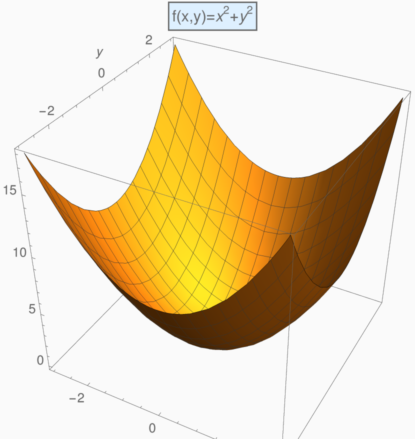
So this is maximized when $\theta = 0$, i.e., when $\mathbf{u}$ and $\nabla f(\mathbf{a})$ are of the same direction.

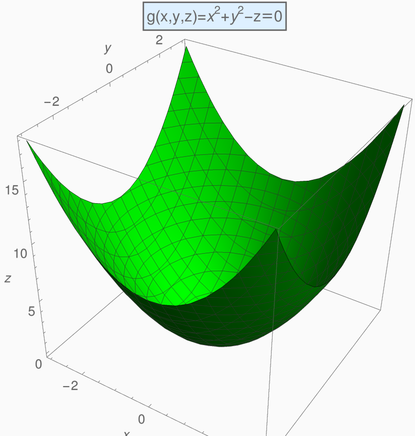
Also, this is minimized when $\theta = \pi$, i.e., when $\mathbf{u}$ and $\nabla f(\mathbf{a})$ are of the opposite direction.

Two ways to view a graph

The graph of a function $f(x, y)$ of two variables can be seen as the level surface $g(x, y, z) = 0$ of the 3-variable function

$$g(x, y, z) = f(x, y) - z$$



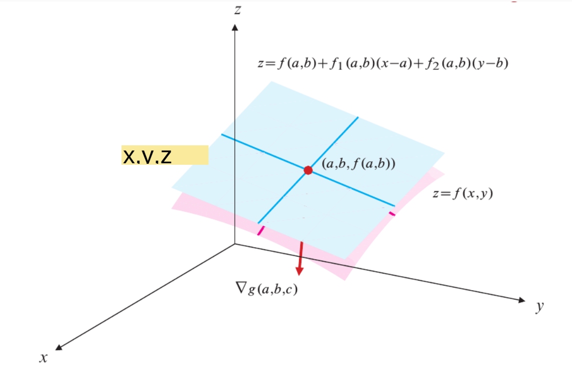


Tangent plane and gradient

Note that $\nabla g(a, b, f(a, b))$ of $g(x, y, z) = f(x, y) - z$ is normal to the tangent plane at $(a, b, f(a, b))$.

So any point (x, y, z) on the tangent plane statisfies

$$(x - a, y - b, z - f(a, b)) \cdot \nabla g(a, b, f(a, b)) = 0$$



Tangent plane and gradient

The gradient of $g(x, y, z) = f(x, y) - z$ at a point $(a, b, f(a, b))$ is

$$\nabla g(a, b, f(a, b)) = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right) = \left(\frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b), -1\right)$$

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Thus the tangent plane of $f(x, y)$ at (a, b) is given by

$$(x - a, y - b, z - f(a, b)) \cdot \nabla g(a, b, f(a, b)) = 0$$

i.e.,

$$\frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) - (z - f(a, b)) = 0.$$

Tangent plane and gradient

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i.e.,

$$\frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) - (z - f(a, b)) = 0.$$

Rearranging the terms gives

$$z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$

Example

Find an equation for the tangent plane at the point $(1, -1, 2)$ of the paraboloid given by the equation

$$x^2 + y^2 - z = 0.$$