Lecture 09 – 12.9 Taylor's Formula, Taylor Series, and Approximations

Several Variable Calculus, 1MA017

Xing Shi Cai

Autumn 2019

Department of Mathematics, Uppsala University, Sweden

Summary

Please watch this video before the lecture: 9

Today we will talk about

- 12.9 Taylor's Formula, Taylor Series, and Approximations

Brook Taylor

Brook Taylor (1685–1731). English mathematician.

Computing without a computer

You are asked to compute $f(1.02, 1.97)$ for $f = \sqrt{x^2 + y^3}$ without a computer. 😟

You can approximate it with the linearisation of $f(x, y)$, i.e.,

$$f(1.02, 1.97) \approx f(1,2) + \frac{\partial f}{\partial x}(1,2)(1.02 - 1) + \frac{\partial f}{\partial y}(1,2)(1.97 - 2)$$
$$= 2.94667.$$

Computing without a computer

You are asked to compute $f(1.02, 1.97)$ for $f = \sqrt{x^2 + y^3}$ without a computer. 😟

You can approximate it with the linearisation of $f(x, y)$, i.e.,

$$\begin{aligned} f(1.02, 1.97) &\approx f(1, 2) + \frac{\partial f}{\partial x}(1, 2)(1.02 - 1) + \frac{\partial f}{\partial y}(1, 2)(1.97 - 2) \\ &= 2.94667. \end{aligned}$$

Questions: How good is this approximation? If we want to be more precise, how can we do it?

Review: Taylor's formula for one-variable

Taylor's formula for one-variable

The Taylor polynomial of degree n for the function $f(x)$ at $x = a$ is

$$p_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor's formula for one-variable

The Taylor polynomial of degree n for the function $f(x)$ at $x = a$ is

$$p_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

We have $p_n(x) \approx f(x)$ near a because

$$r_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(s)}{(n+1)!}(x-a)^{n+1}$$

for some number s between a and x — Taylor's Theorem (Thm. 4.12).

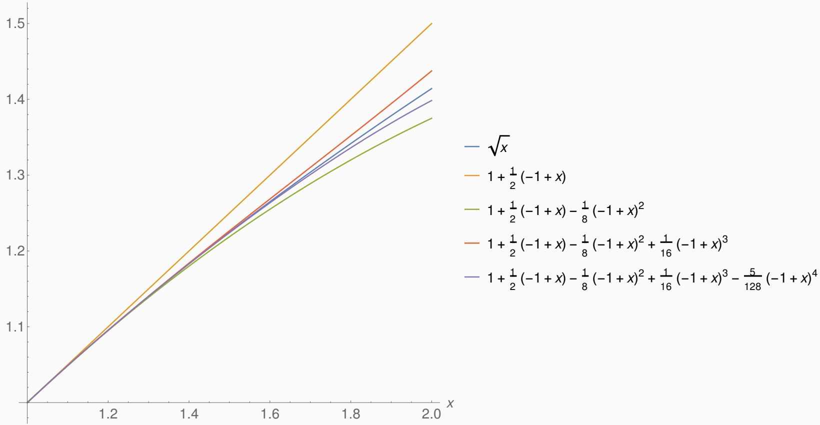
Example — $\sqrt{x}$

For $f(x) = \sqrt{x}$, at the point $x = 1$,

$$
\begin{aligned}
p_1(1.1) =& 1.05000000000000 \\
p_2(1.1) =& 1.04875000000000 \\
&\dots \\
p_9(1.1) =& 1.04880884817101 \\
f(1.1) =& 1.04880884817015
\end{aligned}
$$

Example — $\sqrt{x}$

For $f(x) = \sqrt{x}$, at the point $x = 1$,



Taylor’s formula for several variables

A bit notations

Let $\mathbf{h} = (h_1, \dots h_n)$. For a function $g(x_1, \dots, x_n)$, we define

$$(\mathbf{h} \cdot \nabla) g = \mathbf{h} \cdot \nabla g = (h_1, \dots, h_n) \cdot \left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \dots \frac{\partial g}{\partial x_n} \right)$$

$$= h_1 \frac{\partial g}{\partial x_1} + h_2 \frac{\partial g}{\partial x_2} + \dots + h_n \frac{\partial g}{\partial x_n}$$

A bit notations

Let $\mathbf{h} = (h_1, \dots h_n)$. For a function $g(x_1, \dots, x_n)$, we define

$$(\mathbf{h} \cdot \nabla)g = \mathbf{h} \cdot \nabla g = (h_1, \dots, h_n) \cdot \left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \dots \frac{\partial g}{\partial x_n} \right)$$

$$= h_1 \frac{\partial g}{\partial x_1} + h_2 \frac{\partial g}{\partial x_2} + \dots + h_n \frac{\partial g}{\partial x_n}$$

For example, if $g(x, y) = xy$ and $\mathbf{h} = (2, 1)$, then

$$(\mathbf{h} \cdot \nabla)g = \mathbf{h} \cdot \nabla g = (2, 1) \cdot (y, x) = 2y + x.$$

A bit notations

Let $\mathbf{h} = (h_1, \dots h_n)$. For a function $g(x_1, \dots, x_n)$, we define

$$(\mathbf{h} \cdot \nabla) g = \mathbf{h} \cdot \nabla g = (h_1, \dots, h_n) \cdot \left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \dots \frac{\partial g}{\partial x_n} \right)$$
$$= h_1 \frac{\partial g}{\partial x_1} + h_2 \frac{\partial g}{\partial x_2} + \cdots + h_n \frac{\partial g}{\partial x_n}$$

For example, if $g(x, y) = xy$ and $\mathbf{h} = (2, 1)$, then

$$(\mathbf{h} \cdot \nabla) g = \mathbf{h} \cdot \nabla g = (2, 1) \cdot (y, x) = 2y + x.$$

Since $(\mathbf{h} \cdot \nabla) g$ is again a function of n variables, we can do this repeatedly

$$(h \cdot \nabla)^2 g = (h \cdot \nabla)(h \cdot \nabla) g$$
$$(h \cdot \nabla)^3 g = (h \cdot \nabla)(h \cdot \nabla)(h \cdot \nabla) g$$
$$\dots$$

Taylor's formula for n variables

Given $f : \mathbb{R}^n \mapsto \mathbb{R}$ and $\mathbf{a} = (a_1, \dots, a_n), \mathbf{x} = (x_1, \dots, x_n)$, let $\mathbf{h} = (h_1, \dots, h_n) = \mathbf{x} - \mathbf{a}$ and

$$F(t) = f(\mathbf{y}(t)) = f(\mathbf{a} + t\mathbf{h}).$$

where

$$\mathbf{y}(t) = \mathbf{a}+t\mathbf{h} = (a_1, \dots, a_n)+t(h_1, \dots, h_n) = (a_1+th_1, \dots, a_n+th_n)$$

Taylor's formula for n variables

Given $f : \mathbb{R}^n \mapsto \mathbb{R}$ and $\mathbf{a} = (a_1, \dots, a_n), \mathbf{x} = (x_1, \dots, x_n)$, let $\mathbf{h} = (h_1, \dots, h_n) = \mathbf{x} - \mathbf{a}$ and

$$F(t) = f(\mathbf{y}(t)) = f(\mathbf{a} + t\mathbf{h}).$$

where

$$\mathbf{y}(t) = \mathbf{a} + t\mathbf{h} = (a_1, \dots, a_n) + t(h_1, \dots, h_n) = (a_1 + th_1, \dots, a_n + th_n)$$

Then by the chain rule

$$F'(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{y}(t)) = f'(\mathbf{y}(t))\mathbf{y}'(t) = \begin{bmatrix} f_1(\mathbf{y}(t)) & \dots & f_n(\mathbf{y}(t)) \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$$

$$= \mathbf{h} \cdot \nabla f(\mathbf{y}(t)) = (\mathbf{h} \cdot \nabla) f(\mathbf{y}(t)).$$

Note $f_i = \frac{\partial f}{\partial x_i}$, the partial derivative of f with respect to x_i.

Taylor's formula for n variables

Let $g(\mathbf{x}) = (\mathbf{h} \cdot \nabla) f(\mathbf{x})$. Taking derivative one more time, we get

$$\begin{aligned} F''(t) =& \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{h} \cdot \nabla) f(\mathbf{y}(t)) = \frac{\mathrm{d}}{\mathrm{d}t} g(\mathbf{y}(t)) \\ &= (\mathbf{h} \cdot \nabla)\, g(\mathbf{y}(t)) = (\mathbf{h} \cdot \nabla)^2\, f(\mathbf{y}(t)) \end{aligned}$$

Taylor's formula for n variables

Let $g(\mathbf{x}) = (\mathbf{h} \cdot \nabla) f(\mathbf{x})$. Taking derivative one more time, we get

$$\begin{aligned} F''(t) =& \frac{\mathrm{d}}{\mathrm{d}t} (\mathbf{h} \cdot \nabla) f(\mathbf{y}(t)) = \frac{\mathrm{d}}{\mathrm{d}t} g(\mathbf{y}(t)) \\ &= (\mathbf{h} \cdot \nabla)\, g(\mathbf{y}(t)) = (\mathbf{h} \cdot \nabla)^2\, f(\mathbf{y}(t)) \end{aligned}$$

Taking derivative 3 times, we get

$$F'''(t) = (\mathbf{h} \cdot \nabla)^3 f(\mathbf{y}(t))$$

Taylor's formula for n variables

Let $g(\mathbf{x}) = (\mathbf{h} \cdot \nabla) f(\mathbf{x})$. Taking derivative one more time, we get

$$F''(t) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{h} \cdot \nabla) f(\mathbf{y}(t)) = \frac{\mathrm{d}}{\mathrm{d}t} g(\mathbf{y}(t))$$
$$= (\mathbf{h} \cdot \nabla)\, g(\mathbf{y}(t)) = (\mathbf{h} \cdot \nabla)^2\, f(\mathbf{y}(t))$$

Taking derivative 3 times, we get

$$F'''(t) = (\mathbf{h} \cdot \nabla)^3 f(\mathbf{y}(t))$$

Taking derivative m times, we get

$$F^{(m)}(t) = (\mathbf{h} \cdot \nabla)^m f(\mathbf{y}(t)) = (\mathbf{h} \cdot \nabla)^m f(\mathbf{a} + t\mathbf{h})$$

In particular

$$F^{(m)}(0) = (\mathbf{h} \cdot \nabla)^m f(\mathbf{a})$$

Taylor's formula for n variables

Applying Taylor's Theorem for one variable functions to

$$f(\mathbf{x}) = f(\mathbf{a} + \mathbf{h}) = f(\mathbf{y}(1)) = F(1),$$

we get, for some $\theta \in [0, 1]$,

Taylor's formula for n variables

Applying Taylor's Theorem for one variable functions to

$$f(\mathbf{x}) = f(\mathbf{a} + \mathbf{h}) = f(\mathbf{y}(1)) = F(1),$$

we get, for some $\theta \in [0, 1]$,

$$\begin{aligned}
f(\mathbf{x}) &= F(1) \\
&= F(0) + \frac{F'(0)}{1!}(1-0) + \cdots + \frac{F^{(m)}(0)}{m!}(1-0)^m + \frac{F^{(m+1)}(\theta)}{(m+1)!} \\
&= f(\mathbf{a}) + \frac{(\mathbf{h} \cdot \nabla) f(\mathbf{a})}{1!} + \cdots + \frac{(\mathbf{h} \cdot \nabla)^m f(\mathbf{a})}{m!} \\
&\qquad + \frac{(\mathbf{h} \cdot \nabla)^{m+1} f(\mathbf{a} + \theta \mathbf{h})}{(m+1)!} \\
&= p_m(\mathbf{x}) + r_m(\mathbf{x}, \theta).
\end{aligned}$$

Taylor's formula for n variables

Applying Taylor's Theorem for one variable functions to

$$f(\mathbf{x}) = f(\mathbf{a}+\mathbf{h}) = f(\mathbf{y}(1)) = F(1),$$

we get, for some $\theta \in [0,1]$,

$$\begin{aligned}
f(\mathbf{x}) &= F(1) \\
&= F(0) + \frac{F'(0)}{1!}(1-0) + \cdots + \frac{F^{(m)}(0)}{m!}(1-0)^m + \frac{F^{(m+1)}(\theta)}{(m+1)!} \\
&= f(\mathbf{a}) + \frac{(\mathbf{h}\cdot\nabla)f(\mathbf{a})}{1!} + \cdots + \frac{(\mathbf{h}\cdot\nabla)^m f(\mathbf{a})}{m!} \\
&\qquad + \frac{(\mathbf{h}\cdot\nabla)^{m+1} f(\mathbf{a}+\theta\mathbf{h})}{(m+1)!} \\
&= p_m(\mathbf{x}) + r_m(\mathbf{x},\theta).
\end{aligned}$$

The polynomial $p_m(\mathbf{x})$ is called the m-th degree Taylor polynomial of $f(\mathbf{x})$ at $\mathbf{a}$.

Let $\mathbf{h} = \mathbf{x} - \mathbf{a}$. Let $(\mathbf{h} \cdot \nabla) f = \mathbf{h} \cdot \nabla f$.

Then,

$$\begin{aligned} f(\mathbf{x}) = f(\mathbf{a}) + \frac{(\mathbf{h} \cdot \nabla) f(\mathbf{a})}{1!} + \frac{(\mathbf{h} \cdot \nabla)^2 f(\mathbf{a})}{2!} + \cdots + \frac{(\mathbf{h} \cdot \nabla)^m f(\mathbf{a})}{m!} \\ + \frac{(\mathbf{h} \cdot \nabla)^{m+1} f(\mathbf{a} + \theta \mathbf{h})}{(m+1)!} \\ = p_m(\mathbf{x}) + r_m(\mathbf{x}, \theta), \end{aligned}$$

for some $\theta \in [0, 1]$. The polynomial $p_m(\mathbf{x})$ is called the m-th degree Taylor polynomial of $f(\mathbf{x})$ at $\mathbf{a}$.

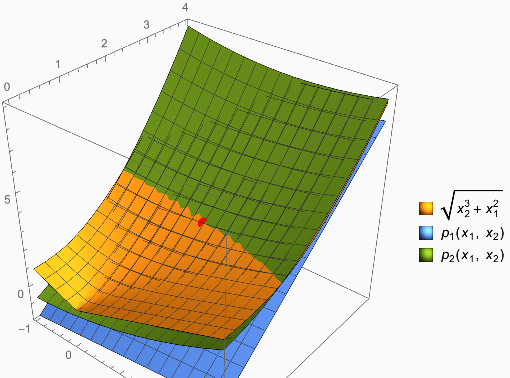
Why do we need Taylor's polynomials?

$r_m(\mathbf{x}, \theta)$ is usually much smaller than $p_m(\mathbf{x})$ for large m, i.e.,

$$f(\mathbf{x}) \approx p_m(\mathbf{x}).$$

This useful for approximating function that is hard to compute.

As shown in the picture, $p_2(x_1, x_2)$ is a pretty good approximation for $f(x_1, x_2) = \sqrt{x_1^2 + x_2^3}$ at $(1, 2)$. We will compute it shortly.



What are $(\mathbf{h} \cdot \nabla)^m f$?

By definition,

$$\begin{aligned}(\mathbf{h} \cdot \nabla) f &= \mathbf{h} \cdot (\nabla f) = (h_1, \dots, h_n) \cdot (f_1, f_2, \dots, f_n) \\ &= h_1 f_1 + \cdots + h_n f_n = \sum_{i=1}^{n} h_i f_i.\end{aligned}$$

What are $(\mathbf{h} \cdot \nabla)^m f$?

By definition,

$$\begin{aligned}(\mathbf{h} \cdot \nabla) f &= \mathbf{h} \cdot (\nabla f) = (h_1, \dots, h_n) \cdot (f_1, f_2, \dots, f_n) \\ &= h_1 f_1 + \dots + h_n f_n = \sum_{i=1}^{n} h_i f_i.\end{aligned}$$

Then

$$\begin{aligned}(\mathbf{h} \cdot \nabla)^2 f &= (\mathbf{h} \cdot \nabla) \sum_{i=1}^{n} h_i f_i = \sum_{i=1}^{n} h_i (\mathbf{h} \cdot \nabla) f_i \\ &= \sum_{i=1}^{n} h_i \sum_{j=1}^{n} h_j f_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} h_i h_j f_{ij}\end{aligned}$$

What are $(\mathbf{h} \cdot \nabla)^m f$?

By definition,

$$(\mathbf{h} \cdot \nabla) f = \mathbf{h} \cdot (\nabla f) = (h_1, \dots, h_n) \cdot (f_1, f_2, \dots, f_n)$$
$$= h_1 f_1 + \cdots + h_n f_n = \sum_{i=1}^{n} h_i f_i.$$

Then

$$(\mathbf{h} \cdot \nabla)^2 f = (\mathbf{h} \cdot \nabla) \sum_{i=1}^{n} h_i f_i = \sum_{i=1}^{n} h_i (\mathbf{h} \cdot \nabla) f_i$$
$$= \sum_{i=1}^{n} h_i \sum_{j=1}^{n} h_j f_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} h_i h_j f_{ij}$$

and similarly

$$(\mathbf{h} \cdot \nabla)^3 f = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} h_i h_j h_k f_{ijk}$$

Note Here $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ and $f_{ijk} = \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}$

Taylor polynomials for two variables

Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{a} = (a_1, a_2)$ and $\mathbf{h} = (h_1, h_2) = \mathbf{x} - \mathbf{a}$. Then for $f(x_1, x_2)$, we have

$$(\mathbf{h} \cdot \nabla) f = \sum_{i=1}^{2} h_i f_i = f_1 h_1 + f_2 h_2,$$

Taylor polynomials for two variables

Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{a} = (a_1, a_2)$ and $\mathbf{h} = (h_1, h_2) = \mathbf{x} - \mathbf{a}$. Then for $f(x_1, x_2)$, we have

$$(\mathbf{h} \cdot \nabla) f = \sum_{i=1}^{2} h_i f_i = f_1 h_1 + f_2 h_2,$$

and

$$\begin{aligned}(\mathbf{h} \cdot \nabla)^2 f &= \sum_{i=1}^{2} \sum_{j=1}^{2} h_i h_j f_{ij} \\ &= f_{11} h_1 h_1 + f_{21} h_2 h_1 + f_{12} h_1 h_2 + f_{22} h_2 h_2 \\ &= f_{11} h_1^2 + 2 f_{12} h_1 h_2 + f_{22} h_2^2,\end{aligned}$$

assuming that f_{21} and f_{12} are continuous.

Taylor polynomials for two variables 😲😲😲😲

Therefore, for two variable functions $f(x_1, x_2)$,

$$p_1(\mathbf{x}) = f(\mathbf{a}) + (\mathbf{h} \cdot \nabla) f(\mathbf{a}) = f(\mathbf{a}) + f_1(\mathbf{a}) h_1 + f_2(\mathbf{a}) h_2$$

which is just the tangent plane of $f(\mathbf{x})$ at $\mathbf{a}$.

Taylor polynomials for two variables 😲😲😲😲

Therefore, for two variable functions $f(x_1, x_2)$,

$$p_1(\mathbf{x}) = f(\mathbf{a}) + (\mathbf{h} \cdot \nabla) f(\mathbf{a}) = f(\mathbf{a}) + f_1(\mathbf{a}) h_1 + f_2(\mathbf{a}) h_2$$

which is just the tangent plane of $f(\mathbf{x})$ at $\mathbf{a}$. And

$$\begin{aligned} p_2(\mathbf{x}) &= f(\mathbf{a}) + (\mathbf{h} \cdot \nabla) f(\mathbf{a}) + \frac{1}{2!} (\mathbf{h} \cdot \nabla)^2 f(\mathbf{a}) \\ &= f(\mathbf{a}) + f_1(\mathbf{a}) h_1 + f_2(\mathbf{a}) h_2 \\ &\quad + \frac{1}{2!} \left(f_{11}(\mathbf{a}) h_1^2 + 2 f_{12}(\mathbf{a}) h_1 h_2 + f_{22}(\mathbf{a}) h_2^2 \right). \end{aligned}$$

where $h_1 = x_1 - a_1$ and $h_2 = x_2 - a_2$.

Example

Qustions: Find $p_2(x_1, x_2)$ for the function $f(x_1, x_2) = \sqrt{x_1^2 + x_2^3}$ at $(1, 2)$.

Example

Qustions: Find $p_2(x_1, x_2)$ for the function $f(x_1, x_2) = \sqrt{x_1^2 + x_2^3}$ at $(1, 2)$.

For this question,

$$\mathbf{a} = (1, 2), \qquad \mathbf{h} = \mathbf{x} - \mathbf{a} = (h_1, h_2) = (x_1 - 1, x_2 - 2).$$

Example

Qustions: Find $p_2(x_1, x_2)$ for the function $f(x_1, x_2) = \sqrt{x_1^2 + x_2^3}$ at $(1, 2)$.

For this question,

$$\mathbf{a} = (1, 2), \qquad \mathbf{h} = \mathbf{x} - \mathbf{a} = (h_1, h_2) = (x_1 - 1, x_2 - 2).$$

So $f(\mathbf{a}) = 3$ and the partial derivatives are

$$f_1(\mathbf{a}) = \frac{1}{3}, \qquad f_2(\mathbf{a}) = 2$$

and

$$f_{11}(\mathbf{a}) = \frac{8}{27}, \qquad f_{12}(\mathbf{a}) = f_{21}(\mathbf{a}) = -\frac{2}{9}, \qquad f_{22}(\mathbf{a}) = \frac{2}{3}.$$

Example – Solution

Putting these numbers into the formula, we get

$$\begin{aligned} p_2(x_1, x_2) &= f(\mathbf{a}) + f_1(\mathbf{a})h_1 + f_2(\mathbf{a})h_2 \\ &\quad + \frac{1}{2!}\left(f_{11}(\mathbf{a})h_1^2 + 2f_{12}(\mathbf{a})h_1h_2 + f_{22}(\mathbf{a})h_2^2\right) \\ &= 3 + \frac{1}{3}(x_1 - 1) + 2(x_2 - 2) \\ &\quad + \frac{1}{2}\left(\frac{8}{27}(x_1 - 1)^2 - \frac{4}{9}(x_1 - 1)(x_2 - 2) + \frac{2}{3}(x_2 - 2)^2\right) \end{aligned}$$

This is a much better approximation for $f(x_1, x_2)$ than $p_1(x_1, x_2)$ (the tangent plane) around $(1, 2)$. For example

$$\begin{aligned} p_1(1.02, 1.97) &= 2.94666666666667 \\ p_2(1.02, 1.97) &= 2.94715925925926 \\ f(1.02, 1.97) &= 2.94716355162044 \end{aligned}$$

Quiz

Which of the following is $p_2(x, y)$ for $f(x, y) = e^{x+2y}$ at $(0, 0)$?

1. $1 + x + 2y + \left(x^2 + 4xy + 4y^2\right)$
2. $1 + x + 2y + \frac{1}{2}\left(x^2 + 4xy + 4y^2\right)$
3. $1 + x + 2y + \frac{1}{2}\left(x^2 + 2xy + 4y^2\right)$
4. $1 + x + 2y + \frac{1}{2}\left(x^2 + 4xy + 2y^2\right)$

Quiz

Which of the following is $p_2(x, y)$ for $f(x, y) = e^{x+2y}$ at $(0, 0)$?

1. $1 + x + 2y + \left(x^2 + 4xy + 4y^2\right)$
2. $1 + x + 2y + \frac{1}{2}\left(x^2 + 4xy + 4y^2\right)$
3. $1 + x + 2y + \frac{1}{2}\left(x^2 + 2xy + 4y^2\right)$
4. $1 + x + 2y + \frac{1}{2}\left(x^2 + 4xy + 2y^2\right)$

Answer

$$p_2(x, y) = 1 + x + 2y + \frac{1}{2}\left(x^2 + 4xy + 4y^2\right)$$