Lecture 10 -- 13.1 Extreme Values

Xing Shi Cai

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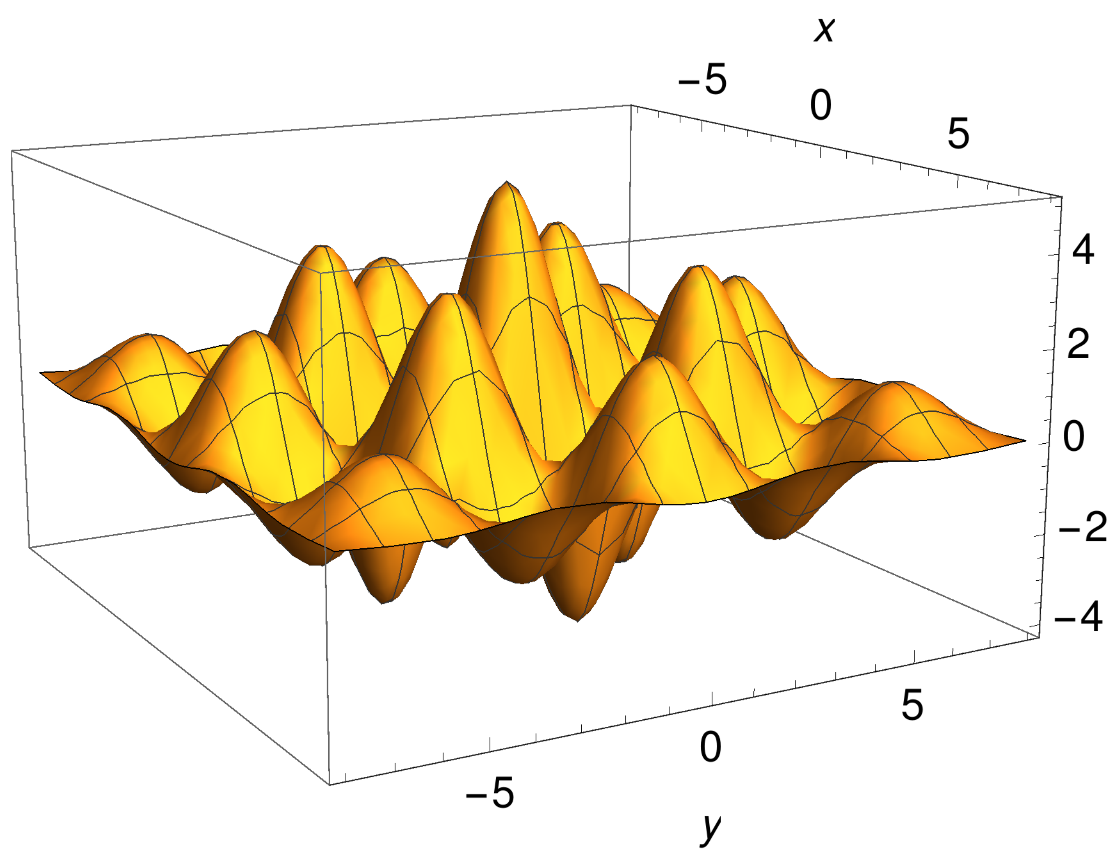
Department of Mathematics, Uppsala University, Sweden

Local maximum and local minium $\mathbb{R}^n \to \mathbb{R}$

A function $f(\boldsymbol{x})$ has a local maximum value (or a local minimum value) at a point a if $f(\boldsymbol{x}) \geq f(\boldsymbol{a})$ (or $f(\boldsymbol{x}) \leq f(\boldsymbol{a})$) for all $\boldsymbol{x}$ close to $\boldsymbol{a}$.

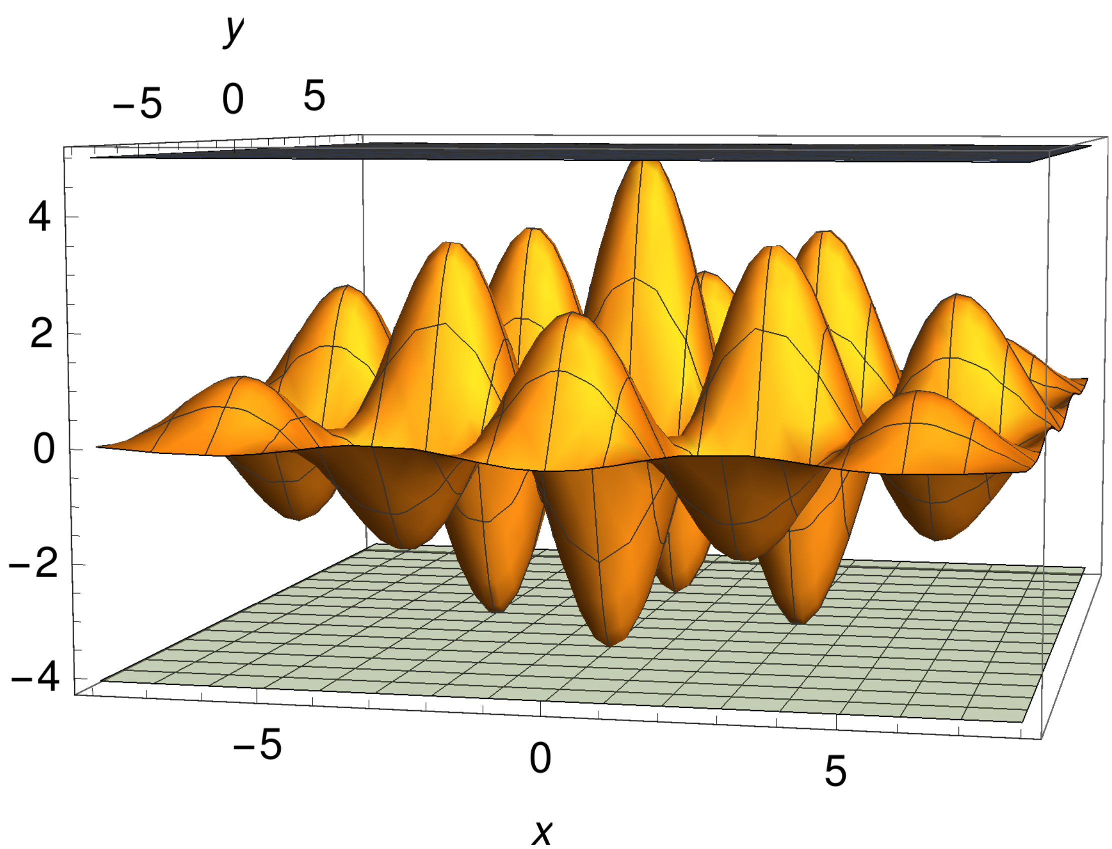
$$5\, e^{\frac{1}{50}\left(-x^2-y^2\right)} \cos(x)\cos(y)$$



Absolute maximum and absolute minium $\mathbb{R}^2 \to \mathbb{R}$

A function $f(\boldsymbol{x})$ has an absolute maximum value (or an absolute minimum value) at a point $\boldsymbol{a}$ domain if $f(\boldsymbol{x}) \geq f(\boldsymbol{a})$ (or $f(\boldsymbol{x}) \leq f(\boldsymbol{a})$) for all $\boldsymbol{x}$ in its domain.

$$5\, e^{\frac{1}{50}\left(-x^2-y^2\right)} \cos(x) \cos(y)$$



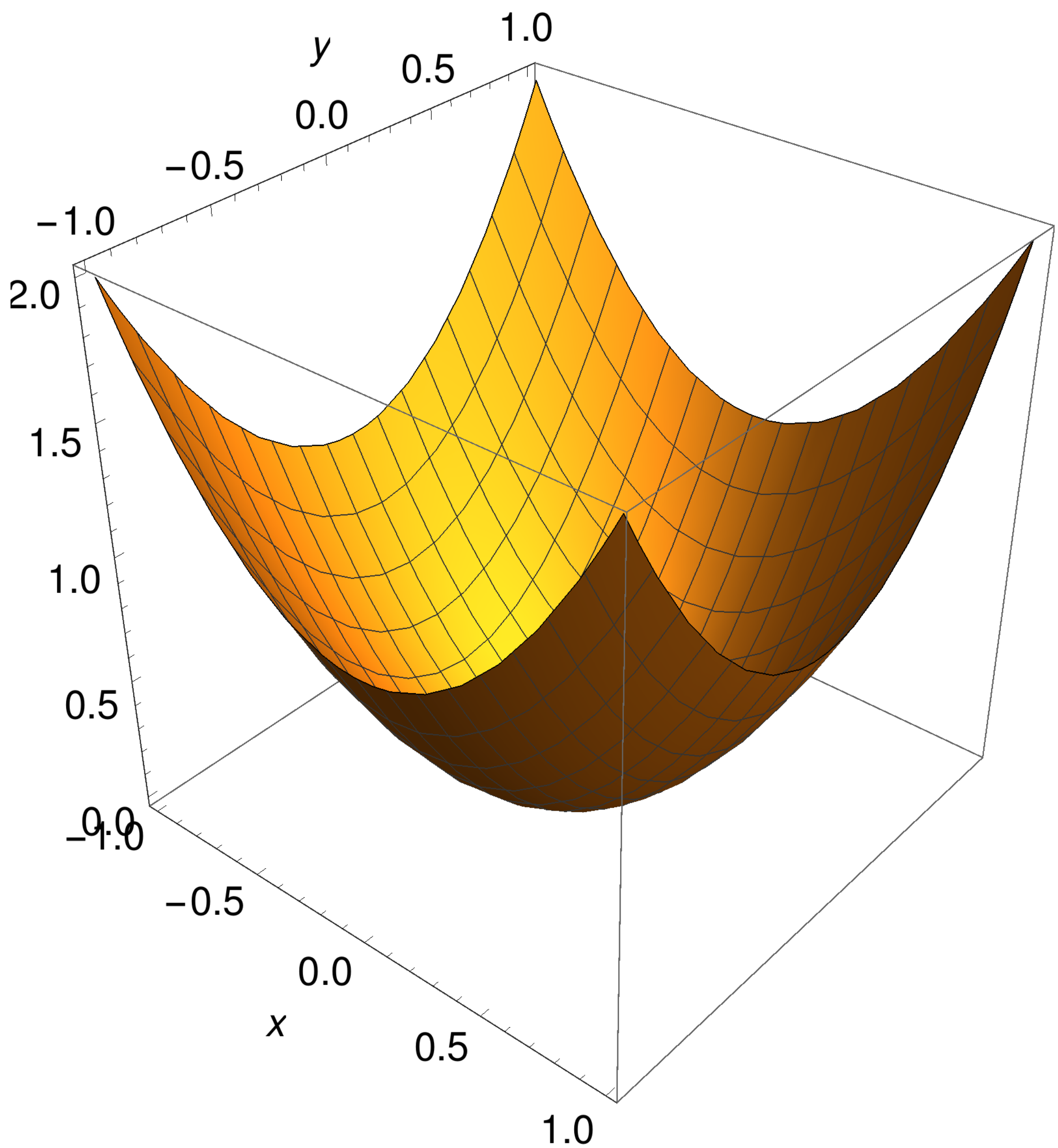
The existence of absolute extreme values

If *f* is a continuous function whose domain is a closed and bounded , then *f* has absolute maximum and minimum values.

Example:

This functions have both absolute maxima and absolute minima.

- $f(x, y) = x^2 + y^2$ defined on $-1 \leq x \leq 1,\ -1 \leq y \leq 1$



Necessary conditions for extreme values

A function $f(x, y)$ can have a local or absolute extreme value at a point (a, b) only if the point is one of:

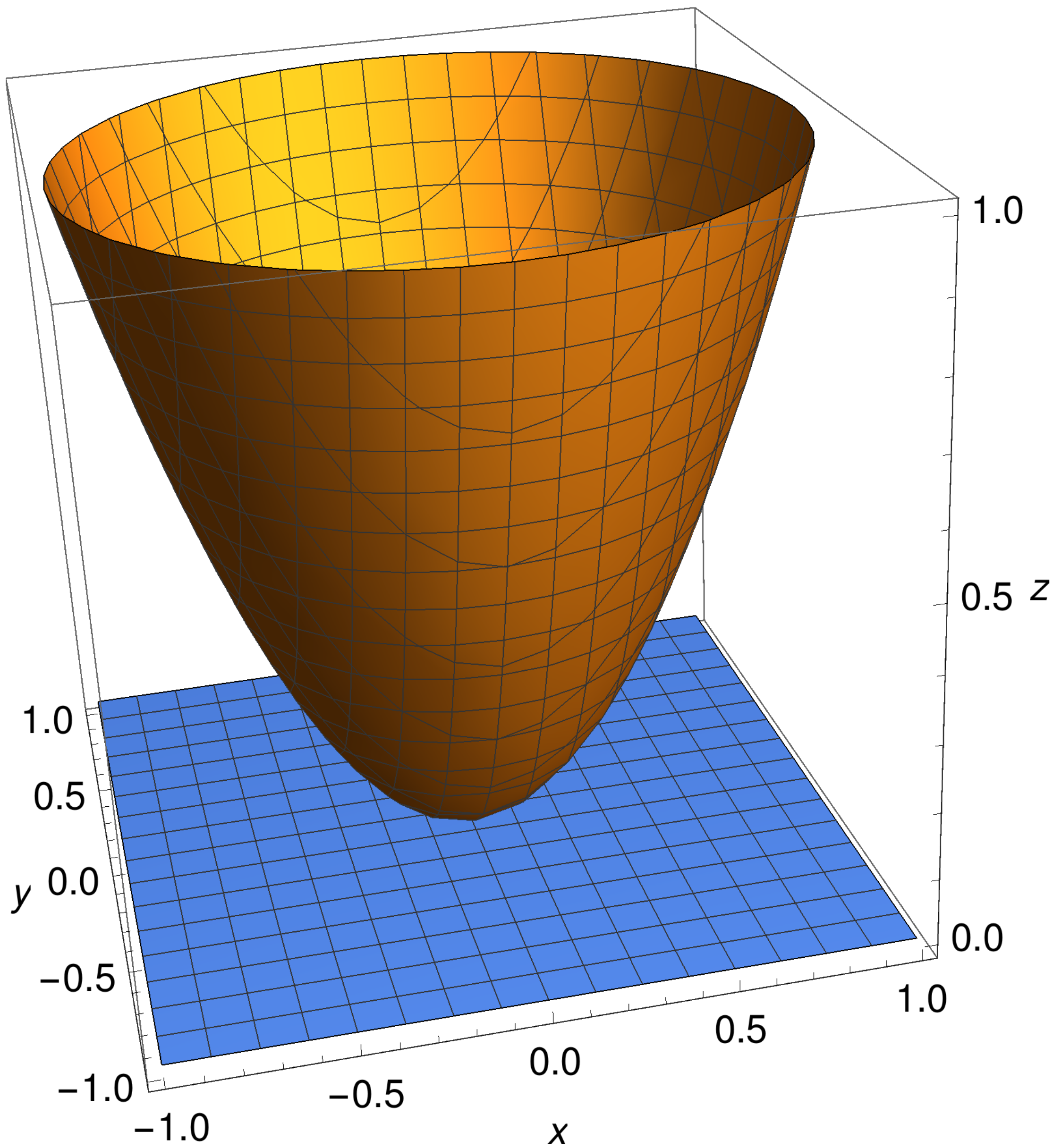
- (a) a critical point -- $\nabla f(a, b) = 0$
- (b) a singular point -- $\nabla f(a, b)$ does **not** exist
- (c) a boundary point

Proof:

- If (a, b) is **not** on the boundary and is not a singular point, then $\nabla f(a, b)$ exists.
- If $\nabla f(a, b) \neq 0$, then f increases along $\nabla f(a, b)$ and decreases along $-\nabla f(a, b)$
- So in this case for $f(a, b)$ to an extreme value, we must have $\nabla f(a, b) = 0$.

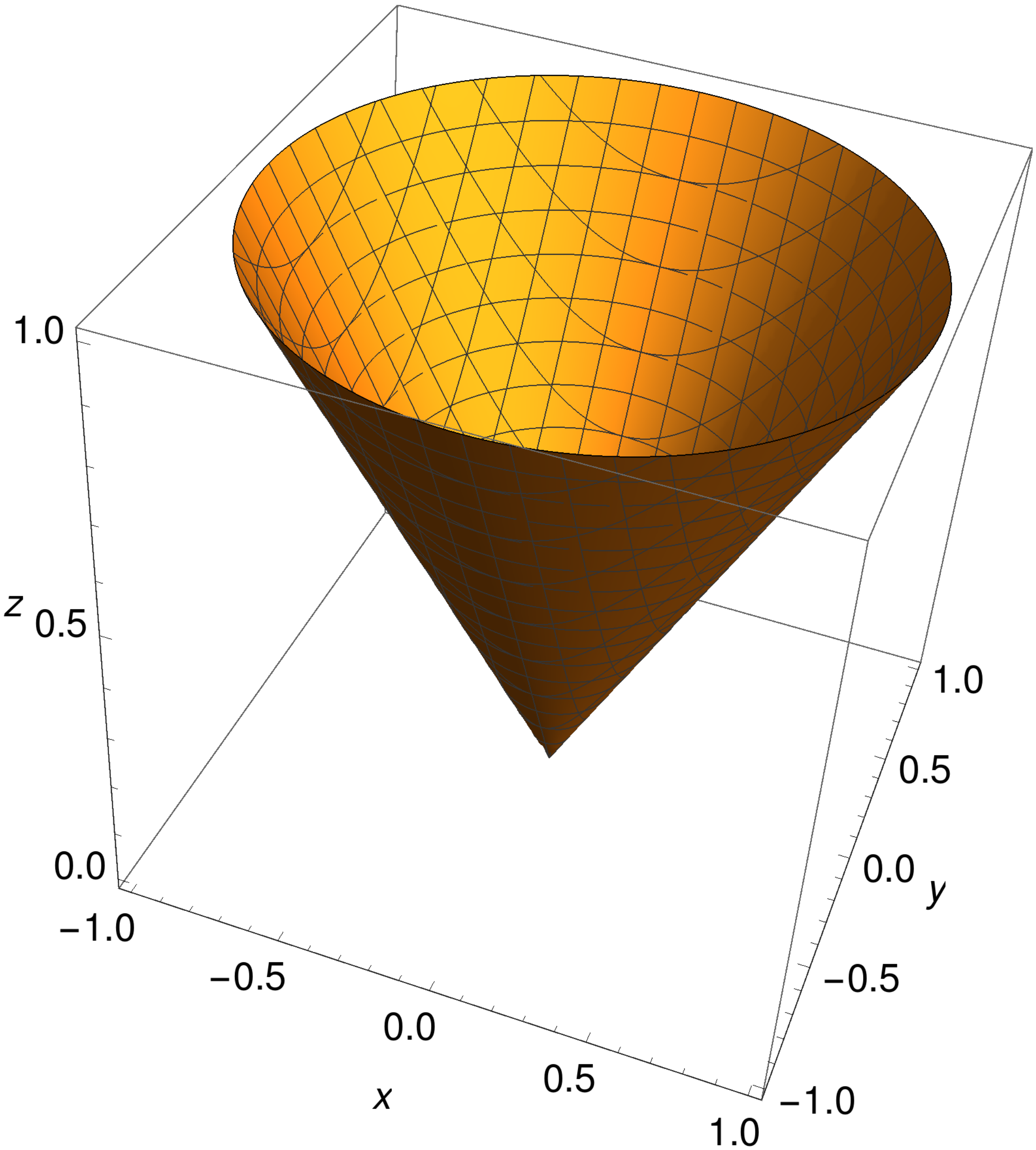
Example — critical point

The function $f(x, y) = x^2 + y^2$ has one critical point at (0, 0). It’s an absolute minimum.



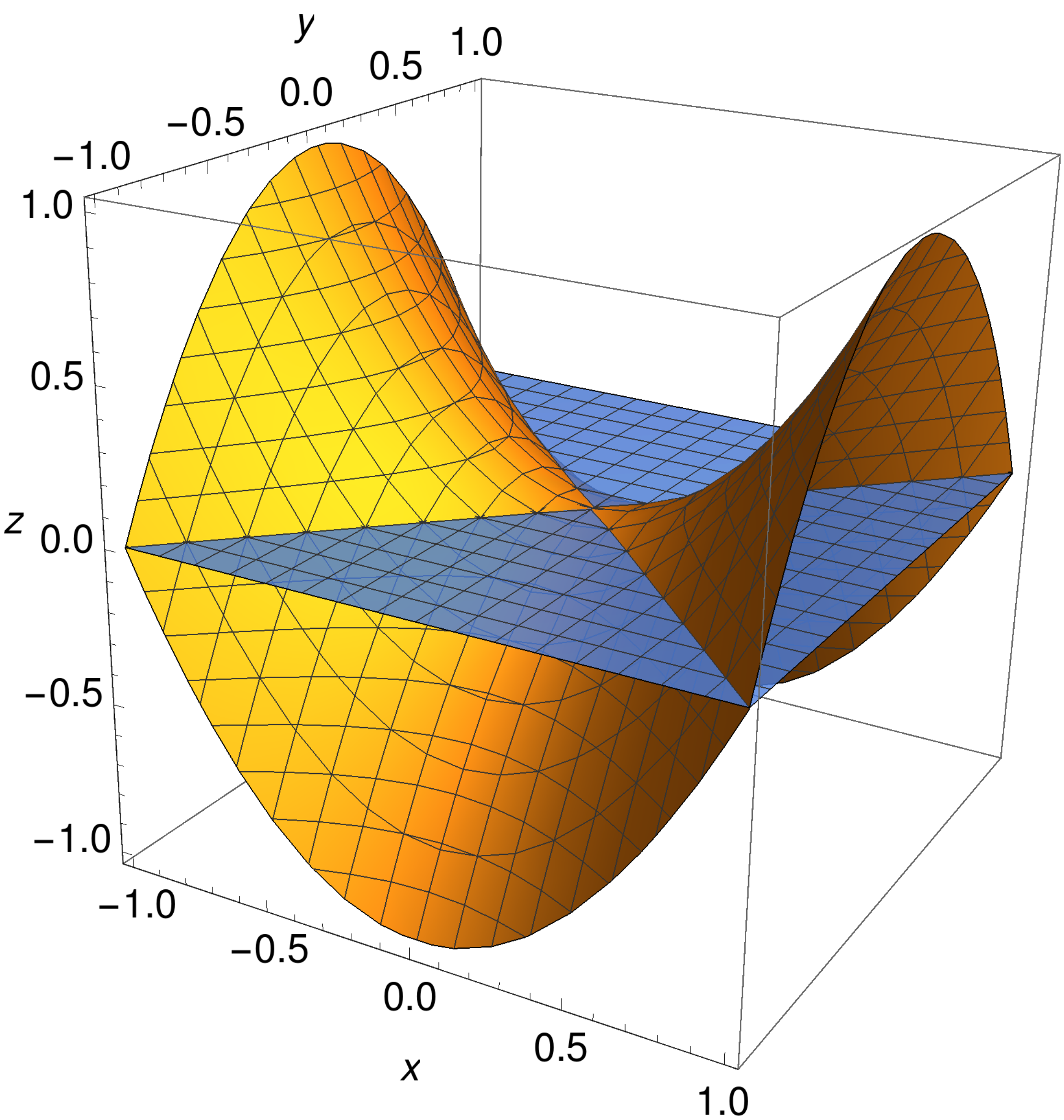
Example — singular point

The function $f(x, y) = \sqrt{x^2 + y^2}$ has one singular point at (0, 0). It's an absolute minimum.



Example — critical point but not extreme point

The function $f(x, y) = x^2 - y^2$ has one critical point at (0, 0). It's neither a minimum nor a maximum. It's called a saddle point.



Classification of critical points

If (a, b) is a critical point of $f(x, y)$, then it can be classified by checking

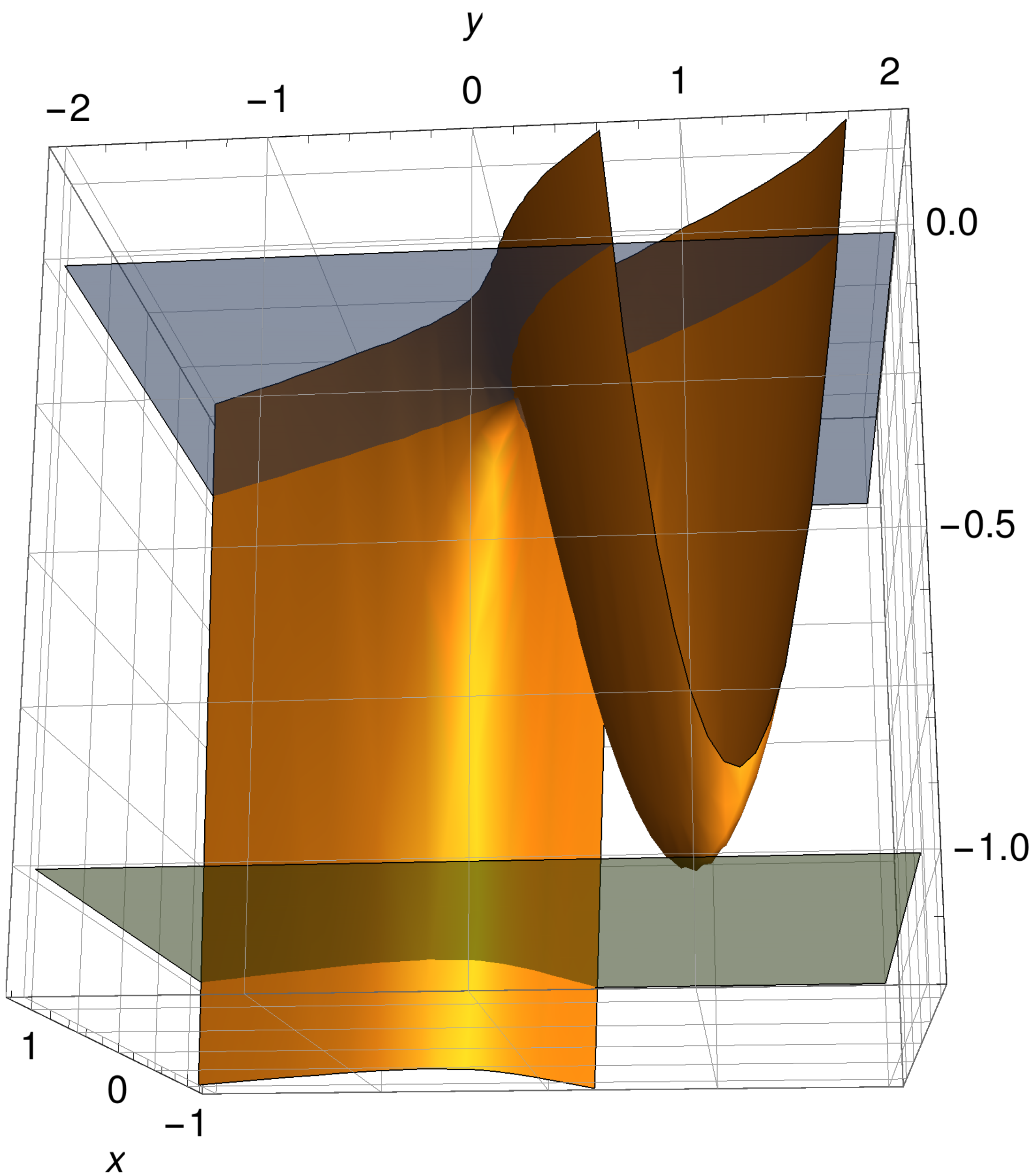
$$\Delta \mathrm{f} = f(a + h, b + k) - f(a, b)$$

If we can find $r > 0$ such that for all $h, k \leq r$

- Δf is always negative, then (a, b) is a local maximum.
- Δf is always positive, then (a, b) is a local minimum.
- Otherwise, (a, b) is a saddle point.

Example

Find and classify the critical points of $f(x, y) = 2x^3 - 6xy + 3y^2$.



A second derivative test — $\mathbb{R}^2 \to \mathbb{R}$

For $f(\boldsymbol{x}) = f(x_1, x_2)$, the Hessian matrix is defined by

$$\mathcal{H}(\boldsymbol{x}) = \begin{pmatrix} f_{11}(\boldsymbol{x}) & f_{12}(\boldsymbol{x}) \\ f_{21}(\boldsymbol{x}) & f_{22}(\boldsymbol{x}) \end{pmatrix}$$

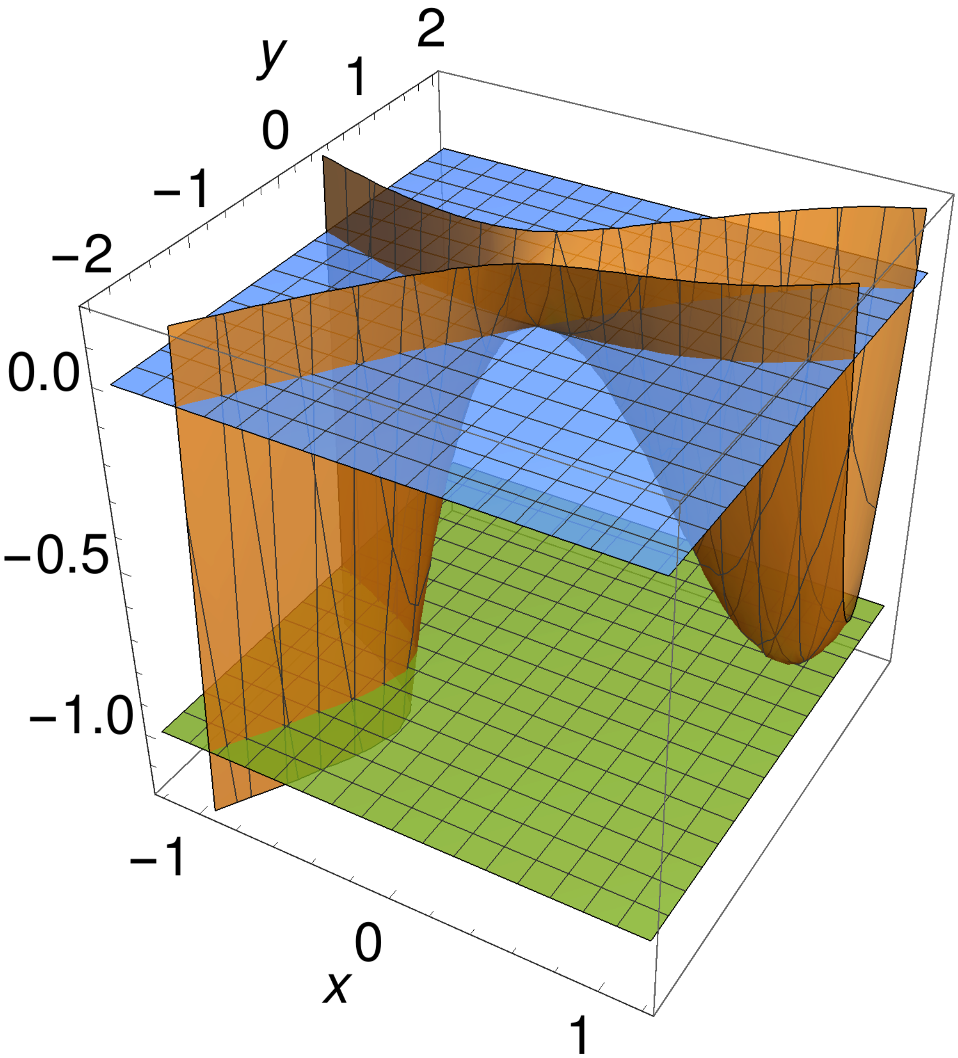
Let $D_1 = f_{11}$ and $D_2 = \det H = f_{11} f_{22} - f_{12} f_{21}$. Then

- (a) If $D_1 > 0$ and $D_2 > 0$, then $\mathcal{H}(\boldsymbol{a})$ is positive definite — minimum.
- (b) If $D_1 < 0$ and $D_2 > 0$, then $\mathcal{H}(\boldsymbol{a})$ is negative definite — maximum.
- (c) If $D_2 < 0$, then $\mathcal{H}(\boldsymbol{a})$ is indefinite — saddle point.
- (d) Otherwise, we know nothing.

(Check Section 10.7 for the definitions.)

Example

The critical points of $f(x, y) = 2x^3 - 6xy + 3y^2$ are (0, 0) and (1,1).



The Hessian matrix of this function is

$$\begin{pmatrix} 12x & -6 \\ -6 & 6 \end{pmatrix}$$

At the two critical points (0, 0) and (1,1), this is

$$\begin{pmatrix} 0 & -6 \\ -6 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 12 & -6 \\ -6 & 6 \end{pmatrix}$$

So (0, 0) is a saddle point and (1, 1) is minimum.