# Lecture 11 — 13.2 Extreme Values of Functions Defined on Restricted Domains

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# Review — A second derivative test — $\mathbb{R}^2 \to \mathbb{R}$

For $f(\boldsymbol{x}) = f(x_1, x_2)$, the Hessian matrix is defined by

$$\mathcal{H}(\boldsymbol{x}) = \begin{pmatrix} f_{11}(\boldsymbol{x}) & f_{12}(\boldsymbol{x}) \\ f_{21}(\boldsymbol{x}) & f_{22}(\boldsymbol{x}) \end{pmatrix}$$

Let $D_1 = f_{11}$ and $D_2 = \det H = f_{11} f_{22} - f_{12} f_{21}$. Then

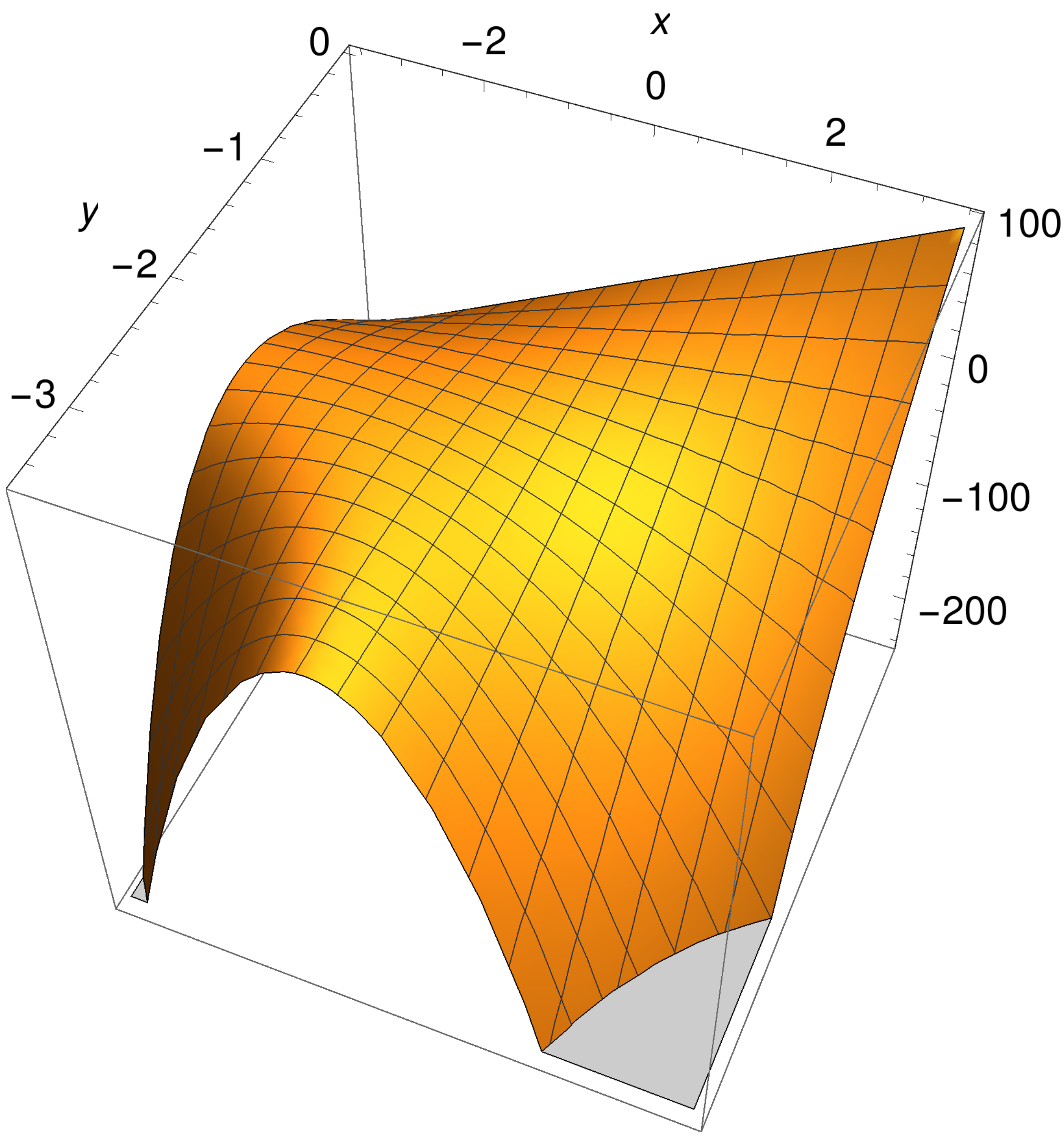
- (a) If $D_1 > 0$ and $D_2 > 0$, then $\mathcal{H}(\boldsymbol{a})$ is positive definite — minimum.
- (b) If $D_1 < 0$ and $D_2 > 0$, then $\mathcal{H}(\boldsymbol{a})$ is negative definite — maximum.
- (c) If $D_2 < 0$, then $\mathcal{H}(\boldsymbol{a})$ is indefinite — saddle point.
- (d) Otherwise, we know nothing.

*See Theorem 8 of Section 10.7*

# Example — A second derivative test — $\mathbb{R}^2 \to \mathbb{R}$

The critical points of $f(x, y) = x^2\, y^3 + 27\, x\, y + 27\, x$ are

$$(-1, -3), (0, -1)$$




The Hessian matrix is

$$\begin{pmatrix} 2\, y^3 & 27 + 6\, x\, y^2 \\ 27 + 6\, x\, y^2 & 6\, x^2\, y \end{pmatrix}$$

At the two critical points, this is

$$\begin{pmatrix} -54 & -27 \\ -27 & -18 \end{pmatrix}, \begin{pmatrix} -2 & 27 \\ 27 & 0 \end{pmatrix}$$

The $D_1$ and $D_2$ are

$$(-54, 243), (-2, -729)$$

So the first point is a maximal point and the second point is a saddle points.

# A second derivative test — $\mathbb{R}^3 \to \mathbb{R}$

For $f(\boldsymbol{x}) = f(x_1, x_2 \ldots, x_n)$, the Hessian matrix is defined by

$$\mathcal{H}(\boldsymbol{x}) = \begin{pmatrix} f_{11}(\boldsymbol{x}) & f_{12}(\boldsymbol{x}) & f_{13}(\boldsymbol{x}) \\ f_{21}(\boldsymbol{x}) & f_{22}(\boldsymbol{x}) & f_{23}(\boldsymbol{x}) \\ f_{31}(\boldsymbol{x}) & f_{32}(\boldsymbol{x}) & f_{33}(\boldsymbol{x}) \end{pmatrix}$$

Let $D_1 = f_{11}$ and $D_2 = f_{11} f_{22} - f_{12} f_{21}$ and $D_3 = \det \mathcal{H}(\boldsymbol{x})$.

- (a) If $D_1 > 0$ and $D_2 > 0$ and $D_3 > 0$, then $\mathcal{H}(\boldsymbol{a})$ is positive definite — minimum.
- (b) If $D_1 < 0$ and $D_2 > 0$ and $D_3 < 0$, then $\mathcal{H}(\boldsymbol{a})$ is negative definite — maximum.
- (c) If $D_3 \neq 0$ but (a) and (b) are false, then $\mathcal{H}(\boldsymbol{a})$ is indefinite — saddle point.
- (d) Otherwise, we know nothing.

*See Theorem 8 of Section 10.7*

# Example — second derivative test — $\mathbb{R}^3 \to \mathbb{R}$

Show that

$$f(x, y, z) = x^2 - 2\,x\,y - 2\,x\,z + 4\,y^2 + 6\,y\,z + 2\,z^2 + 1$$

does not has a local extreme point at (0, 0, 0).

**Solution**: The gradient of the function is

$$\nabla \mathrm{f}(\mathrm{x},\mathrm{y},\mathrm{z}) = (2\,x - 2\,y - 2\,z,\ -2\,x + 8\,y + 6\,z,\ -2\,x + 6\,y + 4\,z)$$

At the point (0, 0, 0)

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(0, 0, 0)
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So (0, 0, 0) is a critical point.

However, the Hessian matrix at this point (actually, at anywhere) is

$$\begin{pmatrix} 2 & -2 & -2 \\ -2 & 8 & 6 \\ -2 & 6 & 4 \end{pmatrix}$$

and $D_1, D_2, D_3$ are

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(2, 12, -8)
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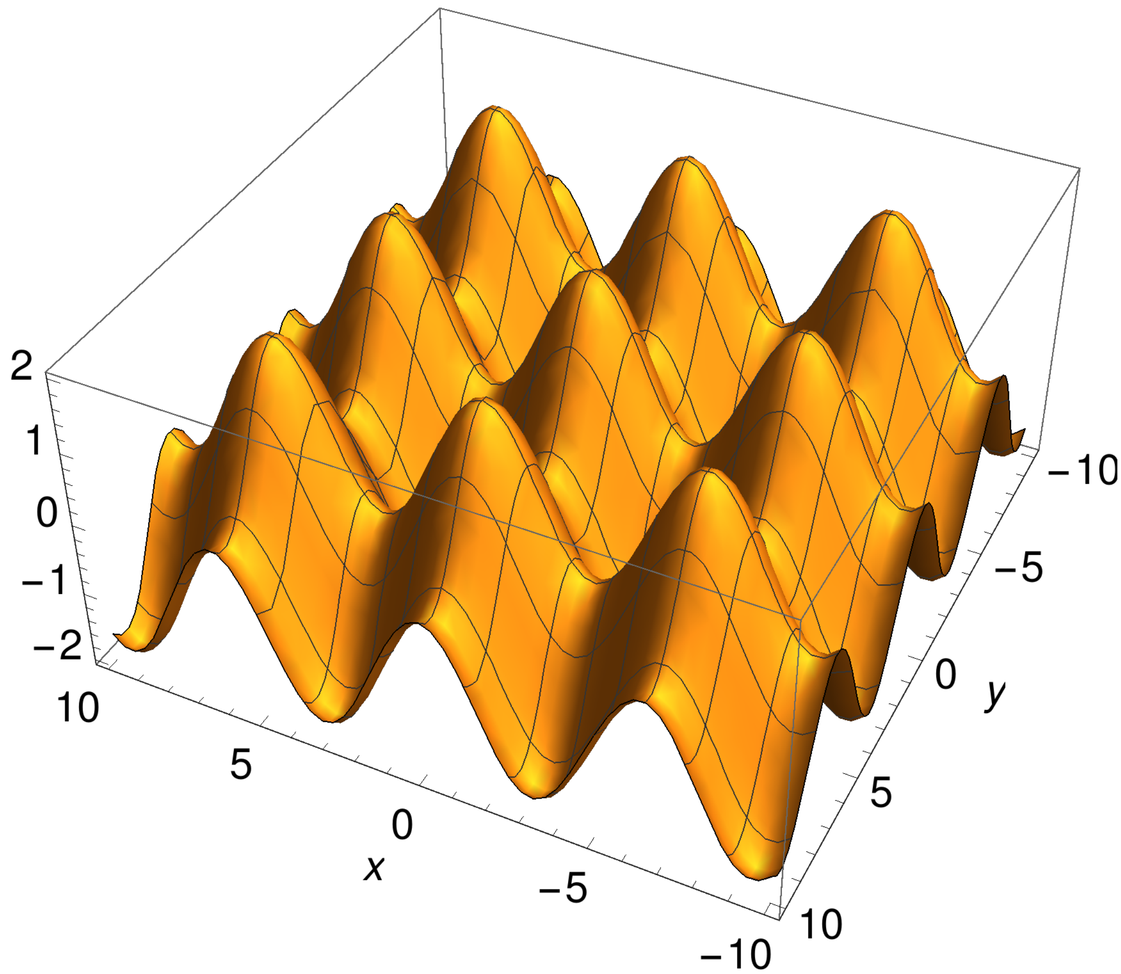
So the matrix is indefinite and we do not have a extreme point.

# Quiz

Find **all** local **minimal** points of $f(x, y) = \cos(x) + \cos(y)$.




## Step 1 — Critical points

The gradient of $f$ is

$$(-\sin(x),\ -\sin(y))$$

So the critical points are $x = m\,\pi,\ \ y = n\,\pi$ for all integers $m, n$.

## Step 2 — Second derivative test

The Hessian matrix of $f$ is

$$\begin{pmatrix} -\cos(x) & 0 \\ 0 & -\cos(y) \end{pmatrix}$$

So $D_1, D_2$ are

$$(-\cos(x),\ \cos(x)\cos(y))$$

When the gradient is 0, when is $D_1 > 0,\ D_2 > 0$ (minimum)?

So the answer is, for $m,\ n \in \mathbb{Z}$,

- $x = 2\,m\,\pi,\ y = 2\,n\,\pi$
- $x = 2\,(m+1)\,\pi,\ y = 2\,(n+1)\,\pi$
- $x = 2\,m\,\pi,\ y = 2\,(n+1)\,\pi$
- $x = 2\,(m+1)\,\pi,\ y = 2\,n\,\pi$

# Finding the absolute extreme value

The existence of absolute max and min requires an argument.

- If the function is continuous and the domain of the function is closed and bounded then there are guaranteed maximum and minimum values.
- Otherwise, they do not necessarily exist and you have to argue they exist.

Absolute maximum and minimum values, if they exist, can be at

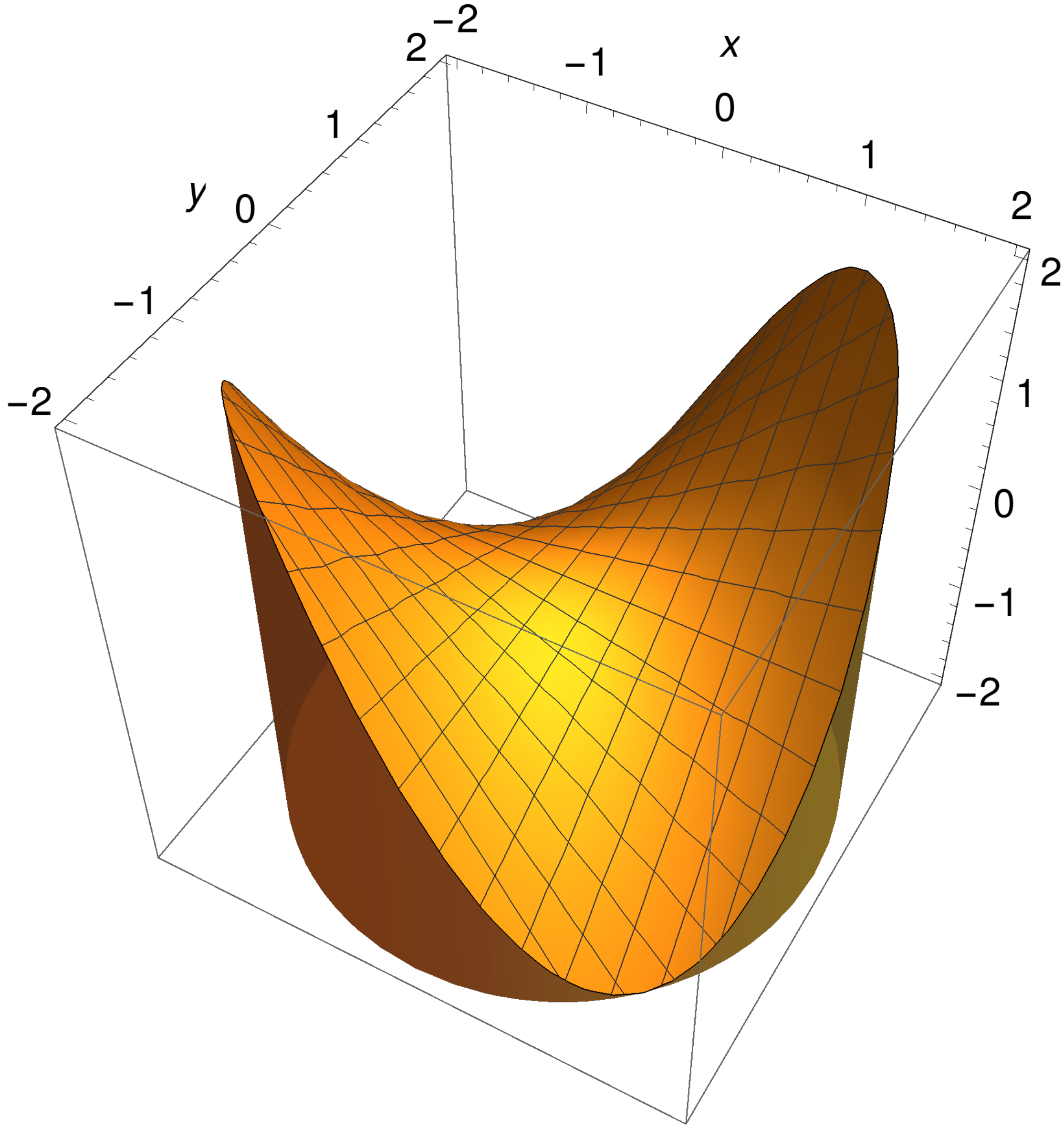
- (a) a critical point -- $\nabla f(a, b) = 0$
- (b) a singular point -- $\nabla f(a, b)$ does **not** exist
- (c) a boundary point

# Example

Determines if $f(x, y) = 2\,x\,y$ have maximal and minimal values for

$$(x, y) \in D = \{(x, y) : x^2 + y^2 \leq 4\}$$

and determine where they are.

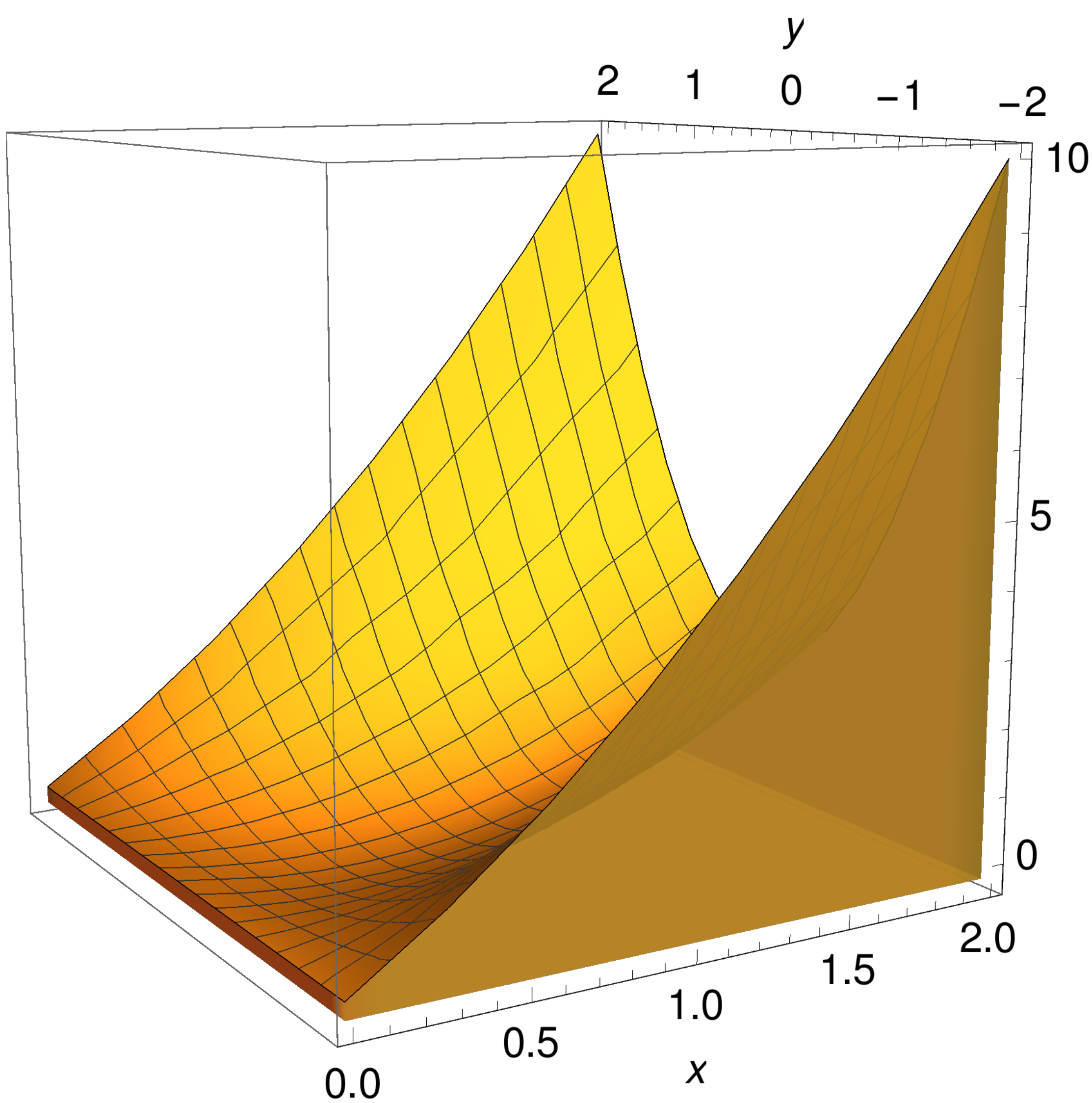


# Example

Determines if $f(x, y) = x^2 + x(y^2 - 1)$ have maximal and minimal values for

$$(x, y) \in D = \{(x, y) : 0 \leq x \leq 2,\ |y| \leq 2\}$$

and determine where they are.



# Example — Solution

Determines if $f(x, y) = x^2 + x(y^2 - 1)$ have maximal and minimal values for

$$(x, y) \in D = \{(x, y) : 0 \leq x \leq 2, \ |y| \leq 2\}$$

and determine where they are.

## Critical points

$\nabla f(x, y)$, the gradient of $f(x, y)$, is

$$(-1 + 2x + y^2, 2xy)$$

The only solutions of $\nabla f(x, y) = (0, 0)$, i.e., the critical points are

$$(0, -1), \left(\frac{1}{2}, 0\right), (0, 1)$$

The Hessian matrix is

$$\begin{pmatrix} 2 & 2y \\ 2y & 2x \end{pmatrix}$$

At the critical points, these are

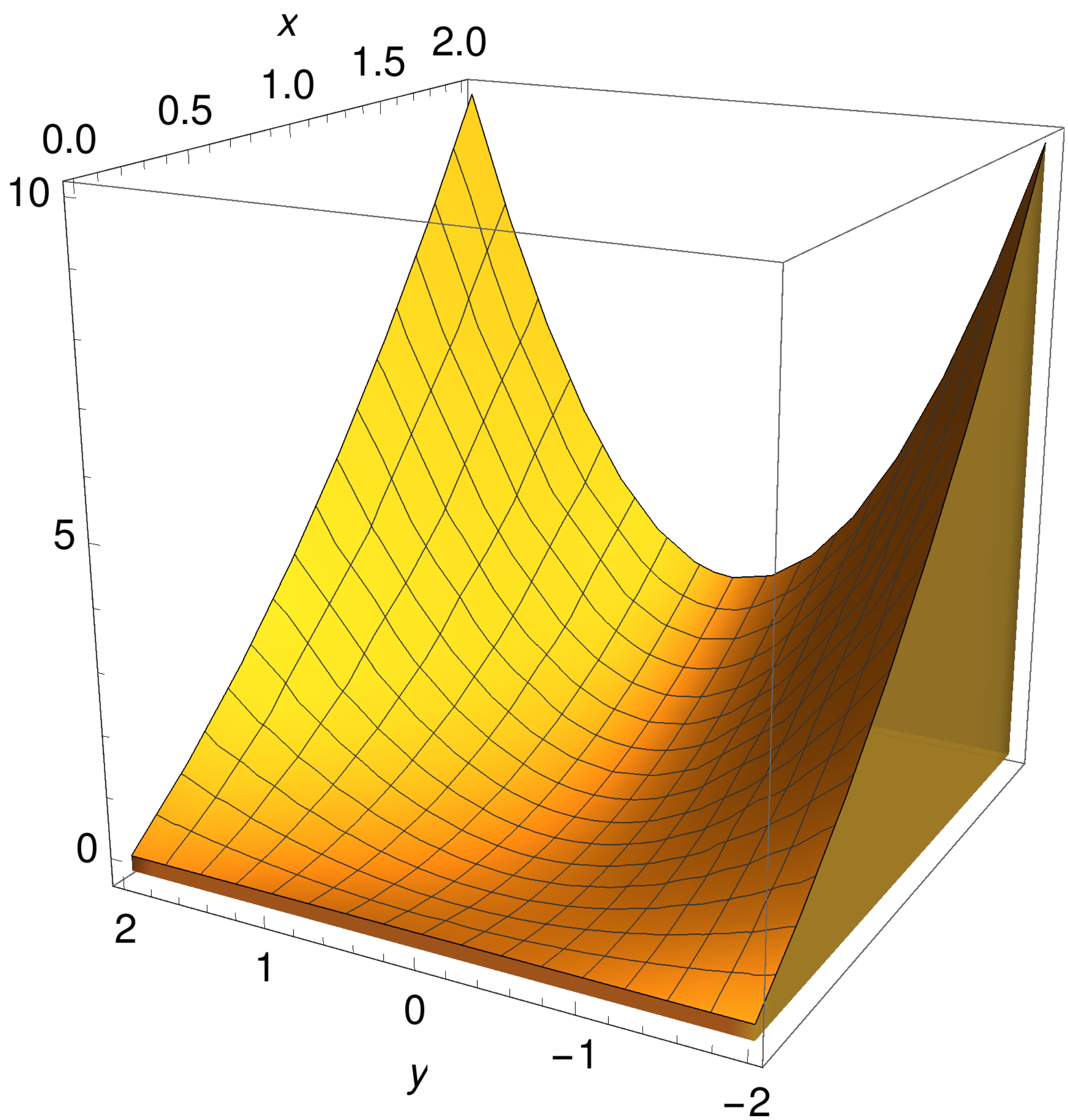
$$\begin{pmatrix} 2 & -2 \\ -2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}$$

The $D_1$ and $D_2$ of each matrix is

$$(2, -4), (2, 2), (2, -4)$$

So only the point $\left(\frac{1}{2}, 0\right)$ is a local minimum with $f\left(\frac{1}{2}, 0\right) = -\frac{1}{4}$. The other two are saddle points.

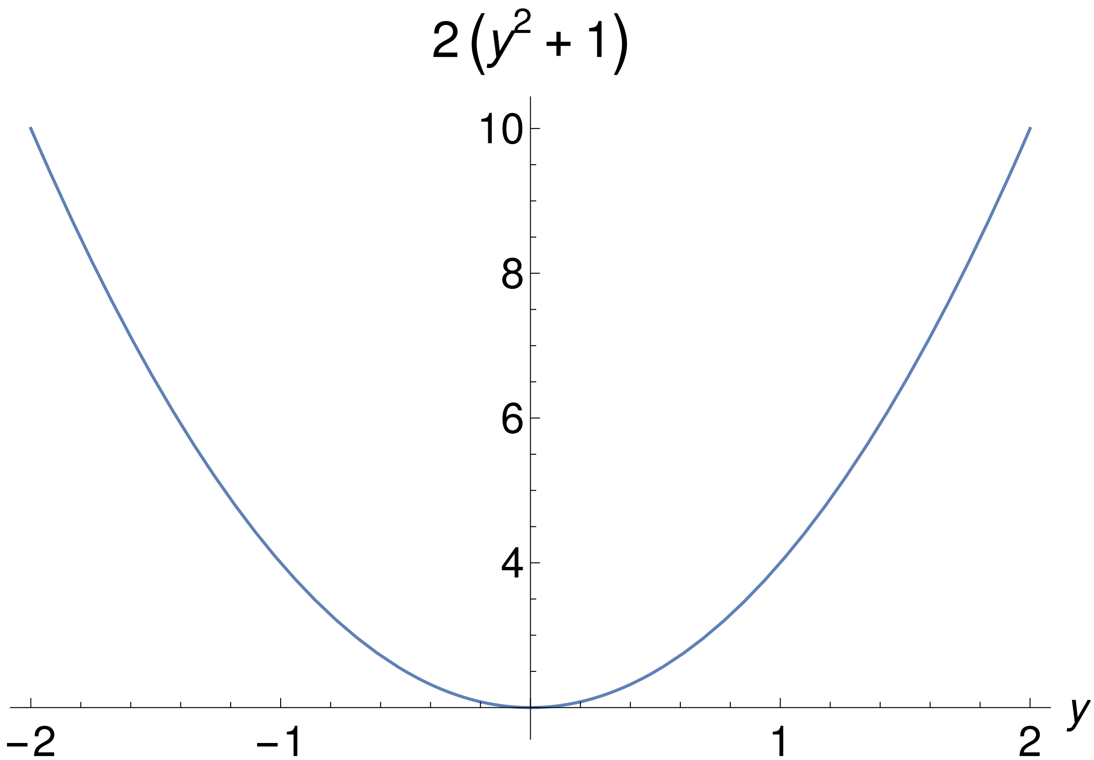
### Boundary points




When $x = 2$ and $-2 \le y \le 2$, $f(x, y)$ is

$$2\left(1 + y^2\right)$$

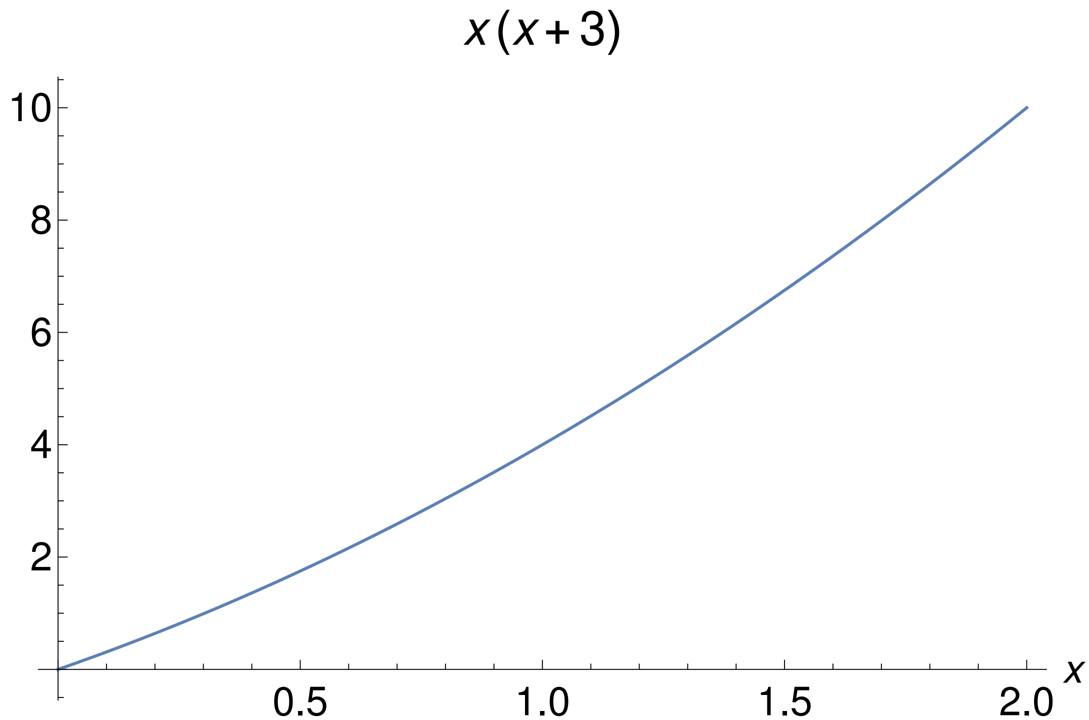
which has maximum value 10 at $y = \pm 2$ and minimal value 2 at $y = 0$.




When $y = \pm 2$ and $0 \leq x \leq 2$, $f(x, y)$ is

$$x (3 + x)$$

which has maximum value 10 at $x = 2$ and minimal value 0 at $x = 0$.




When $x = 0$, $f(x, y) = 0$. So no local extreme points here.

Comparing all the local extreme points we found, we have one absolute minimal point $\left(\frac{1}{2}, 0\right)$ and two absolute maximal points at $(2, \pm 2)$.

# Second derivative test in general ��

For $f(\boldsymbol{x}) = f(x_1, x_2 \ldots, x_n)$, the Hessian matrix is defined by

$$\mathcal{H}(\mathbf{x})=\begin{pmatrix} f_{11}(\boldsymbol{x}) & f_{12}(\boldsymbol{x}) & \ldots & f_{1n}(\boldsymbol{x}) \\ f_{21}(\boldsymbol{x}) & f_{22}(\boldsymbol{x}) & \ldots & f_{2n}(\boldsymbol{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}(\boldsymbol{x}) & f_{n2}(\boldsymbol{x}) & \ldots & f_{nn}(\boldsymbol{x}) \end{pmatrix}$$

- (a) If $\mathcal{H}(\boldsymbol{a})$ is positive definite, then a local minimum at $\boldsymbol{a}$.
- (b) If $\mathcal{H}(\boldsymbol{a})$ is negative definite, then a local maximum at $\boldsymbol{a}$.
- (c) If $\mathcal{H}(\boldsymbol{a})$ is indefinite, then a saddle point at $\boldsymbol{a}$.
- (d) Otherwise, we know nothing.

A matrix $\mathcal{A}$ is

- (a) positive definite if $z^T \mathcal{A} z > 0$ for all $n$-dimensional column vector $z$.
- (b) negative definite if $z^T \mathcal{A} z < 0$ for all $n$-dimensional column vector $z$.
- (c) indefinite if $z^T \mathcal{A} z < 0 < x^T \mathcal{A} x$ for some $n$-dimensional column vector $x$, $z$.

# A bit theory — Why does second derivative test work?

By Taylor's formula for *n*-variable functions

$$f(\boldsymbol{a}+\boldsymbol{h}) = f(\boldsymbol{a}) + (h\cdot\nabla) f(\boldsymbol{a}) + (h\cdot\nabla)^2 f(\boldsymbol{a}+\theta\,\boldsymbol{h})$$

$$= f(a) + \sum_{i=1}^{n} h_i f_i(\boldsymbol{a}) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} h_i h_j f_{ij}(\boldsymbol{a}+\theta\,\boldsymbol{h})$$

$$= f(\boldsymbol{a}) + h\cdot\nabla f(\boldsymbol{a}) + \frac{1}{2}\boldsymbol{h}^T \mathcal{H}(\boldsymbol{a}+\theta\,\boldsymbol{h})\,\boldsymbol{h}$$

Since $\boldsymbol{a}$ is a critical point, $\nabla f(\boldsymbol{a}) = 0$. When $\boldsymbol{h}$ is small $\mathcal{H}(\boldsymbol{a}+\theta\,\boldsymbol{h}) \approx \mathcal{H}(\boldsymbol{a})$. So for $\boldsymbol{h}$ small,

$$f(\boldsymbol{a}+\boldsymbol{h}) - f(\boldsymbol{a}) \approx \frac{1}{2}\boldsymbol{h}^T \mathcal{H}(\boldsymbol{a})\,\boldsymbol{h} \begin{cases} > 0 & \text{if } \mathcal{H}(\mathbf{a}) \text{ is positive definite} \\ < 0 & \text{if } \mathcal{H}(\boldsymbol{a}) \text{ is negative definite} \\ > 0, \ < 0 & \text{if } \mathcal{H}(\boldsymbol{a}) \text{ is indefinite} \end{cases}$$