Lecture 12 —
13.3 Lagrange Multipliers

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Finding the absolute extreme value

Maximum and minimum value, if any, can only be found at

- (a) a critical point -- $\nabla f(a, b) = 0$
- (b) a singular point -- $\nabla f(a, b)$ does **not** exist
- (c) a boundary point

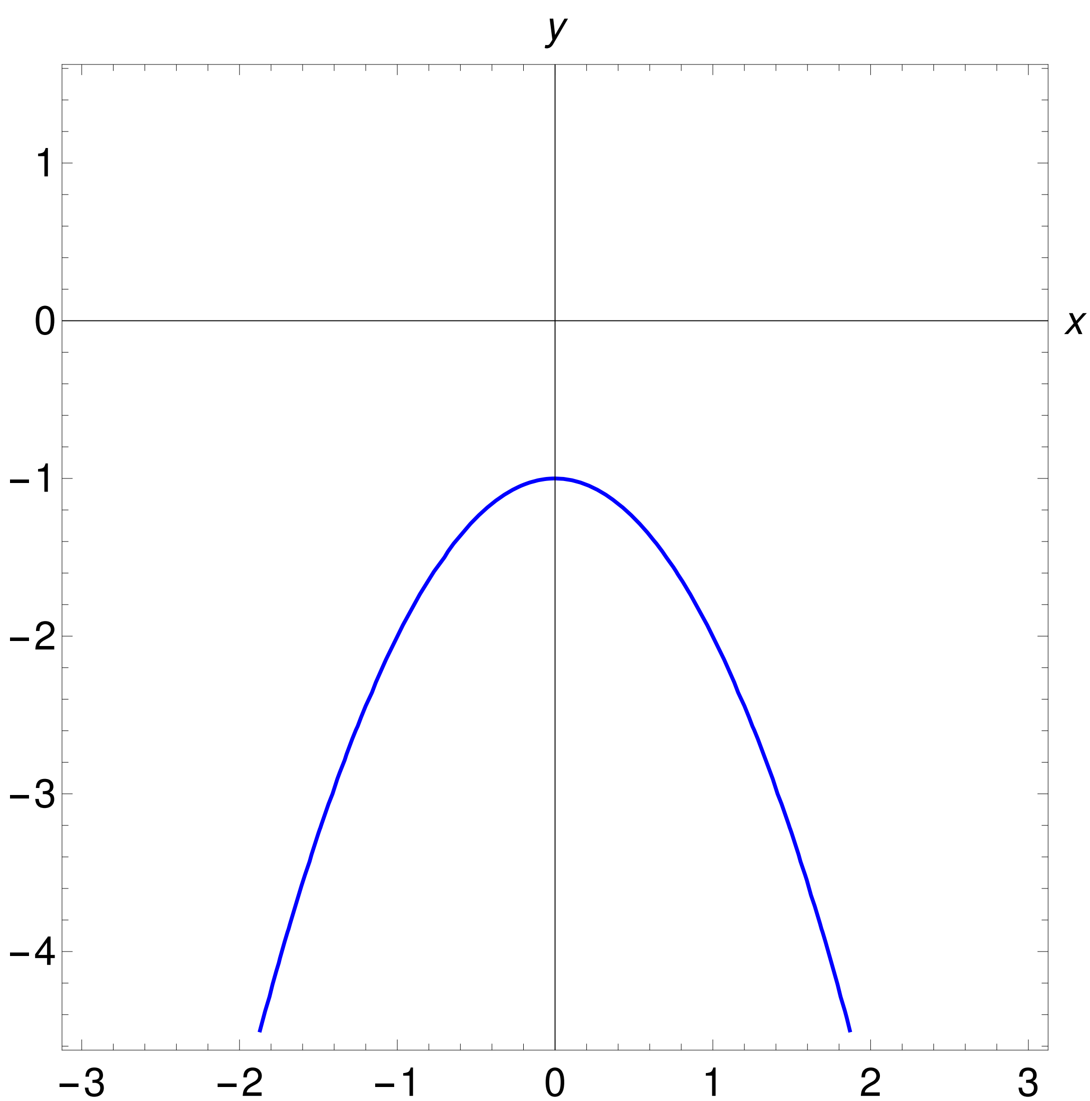
We have to check all of them to find absolute extreme values.

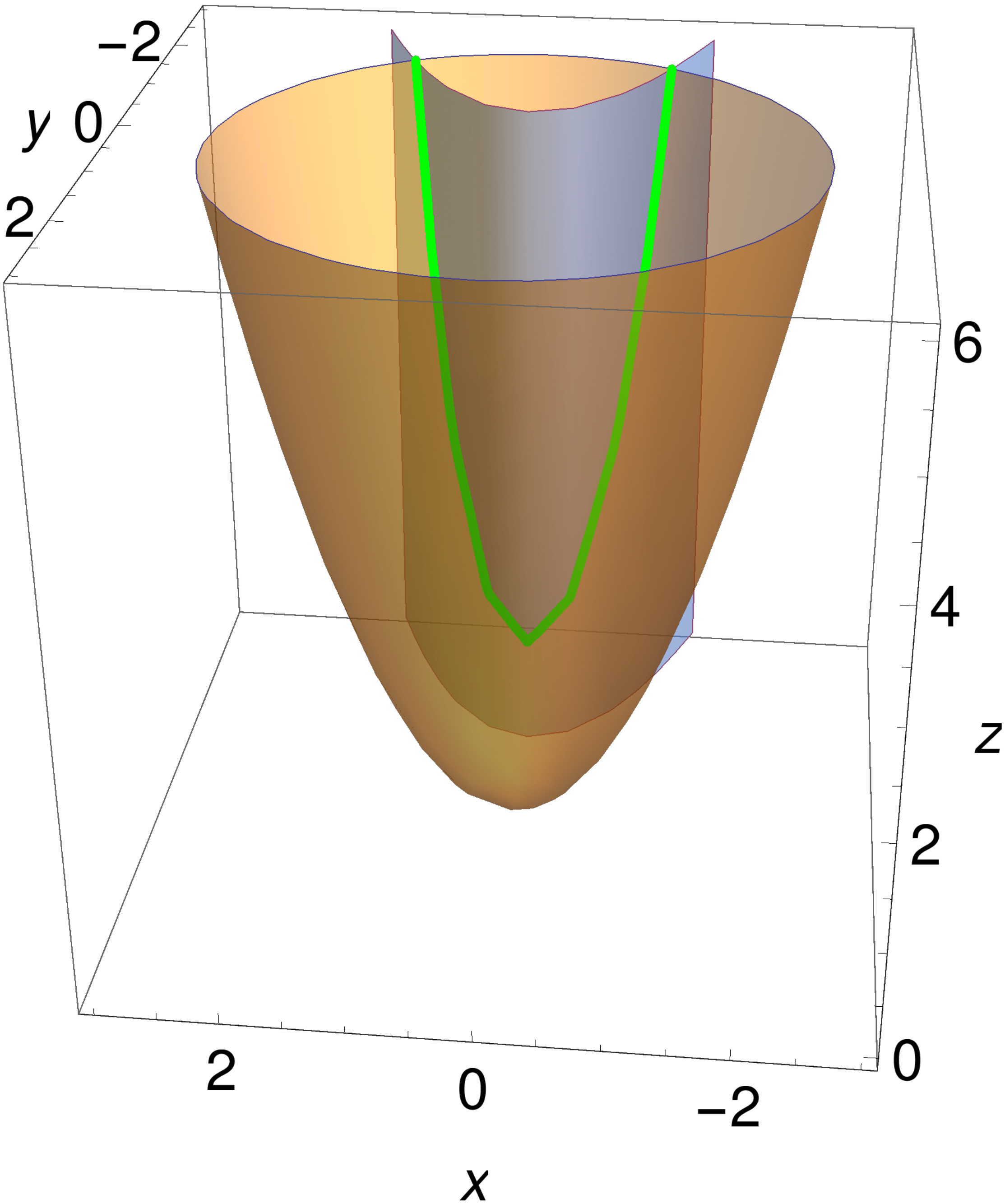
What can we do if the boundary is given by a curve $g(x, y) = 0$?

Finding extreme values with constraint

We may want to find the maximal and minimal values of $f(x, y)$ when (x, y) are restricted to the curve $g(x, y) = 0$.

Example Find the minimal value of $f(x, y) = x^2 + y^2$ for (x, y) on the curve $g(x, y) = y + x^2 + 1 = 0$.





−2
y 0
2
6
4
z
2
0
2
0
−2
x

Minimal value and gradients

It's easy to see from the picture that on the curve $g(x, y) = y + x^2 + 1 = 0$, $f(x, y) = x^2 + y^2$ reaches minimal value at $P_0 = (0, -1)$.

The gradients $\nabla f(x, y)$ and $\nabla g(x, y)$ at P_0 are

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(0, -2), (0, 1)
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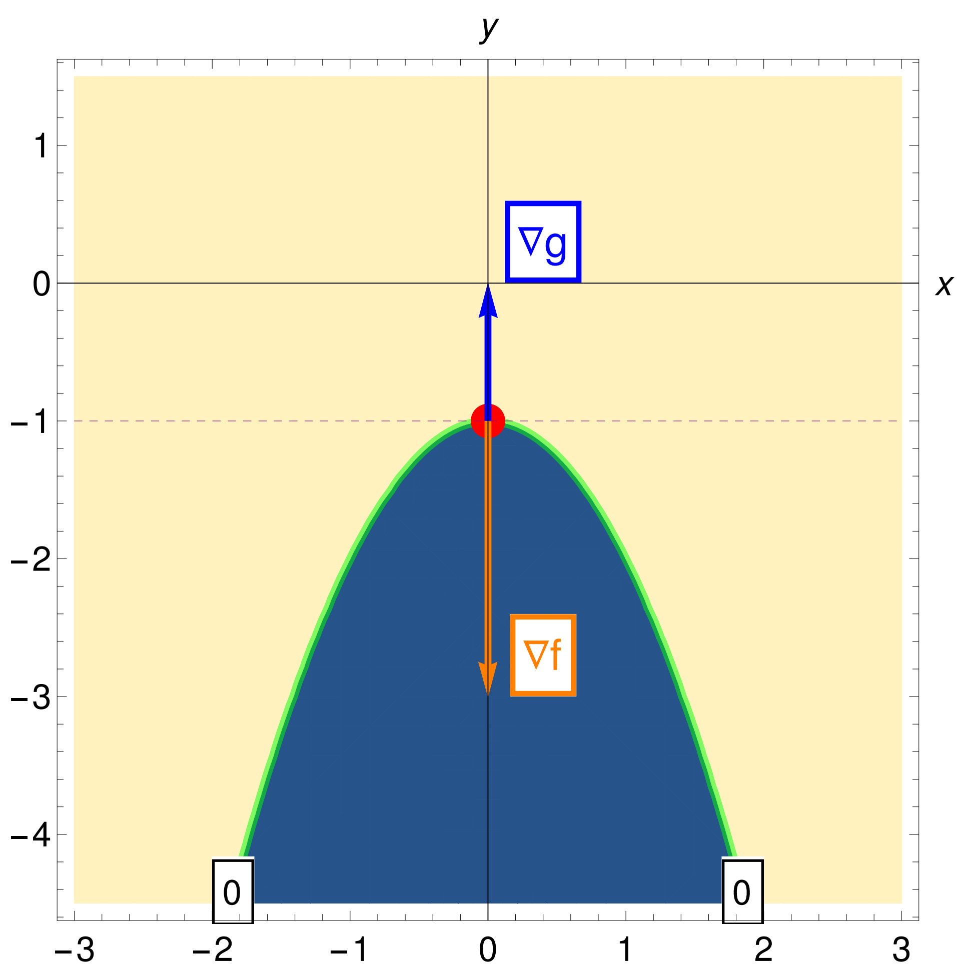
Note that both $\nabla f(x, y)$ and $\nabla g(x, y)$ are orthogonal to the level curve $g(x, y) = 0$, i.e., they are parallel.

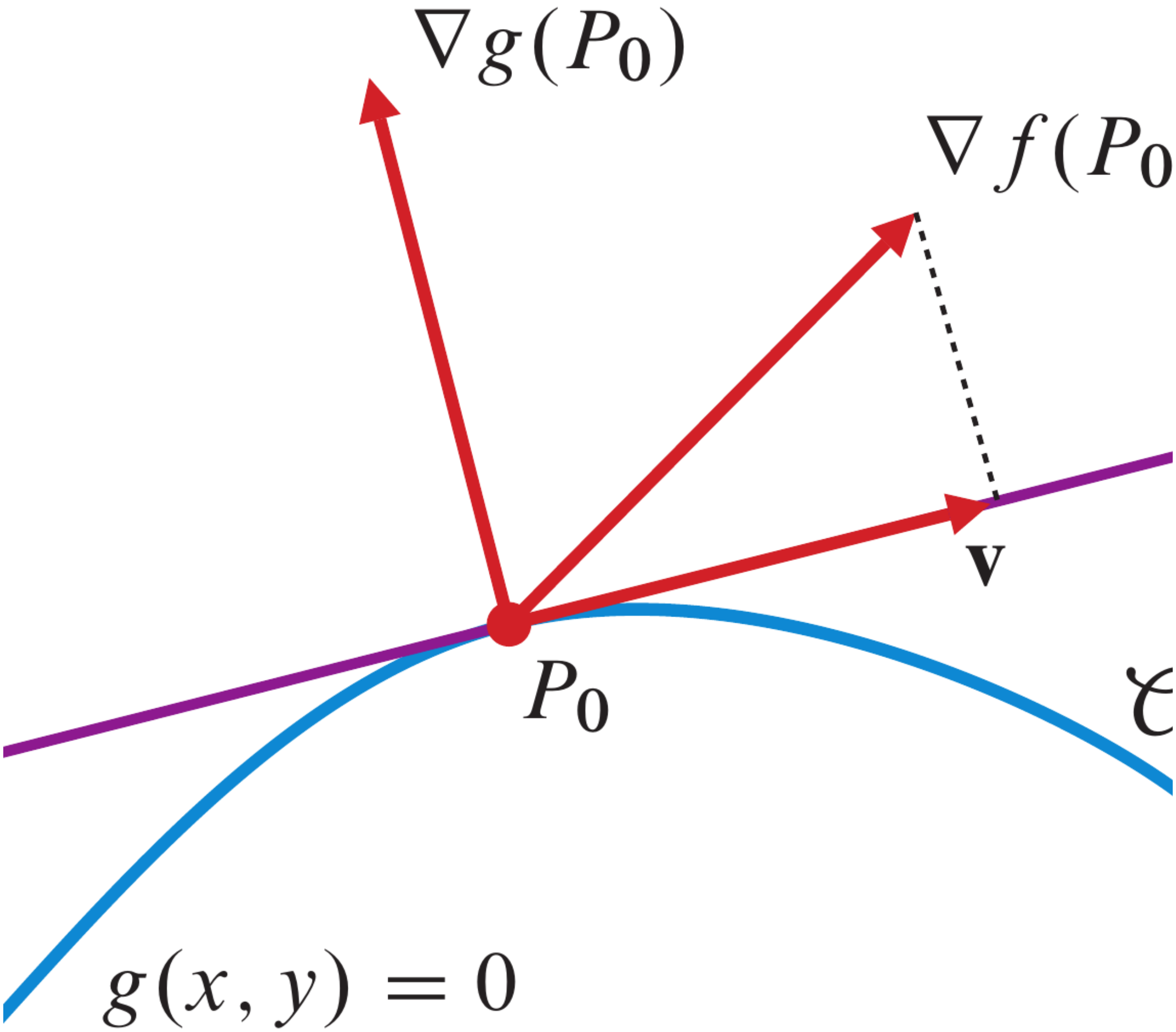


If ∇ $g(P_0)$ and ∇ $f(P_0)$ are not parallel

This means ∇ $f(P_0)$ has a non-zero projection $\boldsymbol{v}$ to the tangent line.

- Along the direction of $\boldsymbol{v}$, $f(x, y)$ increases .
- Along the direction of $-\boldsymbol{v}$, $f(x, y)$ decreases.

This is impossible since $f(P_0)$ is an extreme point on $g(x, y) = 0$.



The Lagrange Multipliers Theorem

Assume that on the curve $g(x, y) = 0$, $f(x, y)$ has an minimal or maximal value at $P_0 = (x_0, y_0)$.

If P_0 is **not** an endpoint of the curve and $\nabla g(x_0, y_0) \neq 0$, then there exists a critical point (x_0, y_0, λ_0) for the Lagrange function

$$L(x, y, \lambda) = f(x, y) + \lambda g(x, y)$$

Proof: Since $\nabla g(x_0, y_0)$ and $\nabla f(x_0, y_0)$ are parallel, there exists $\lambda_0 \neq 0$ such that

$$\nabla f(x_0, y_0) - \lambda_0 \nabla g(x_0 - y_0) = 0.$$

Using this, we can easily verify that $\nabla L(x_0, y_0, \lambda_0) = 0$.

The Lagrange Multipliers Method

To find the extreme points of $f(x, y)$ restricted to the curve $g(x, y) = 0$, we can

- Find the critical points of

$$L(x, y, \lambda) = f(x, y) + \lambda\, g(x, y)$$

- Check if these points are maximal or minimal points
- Check if the end points of the curve are maximal or minimal points

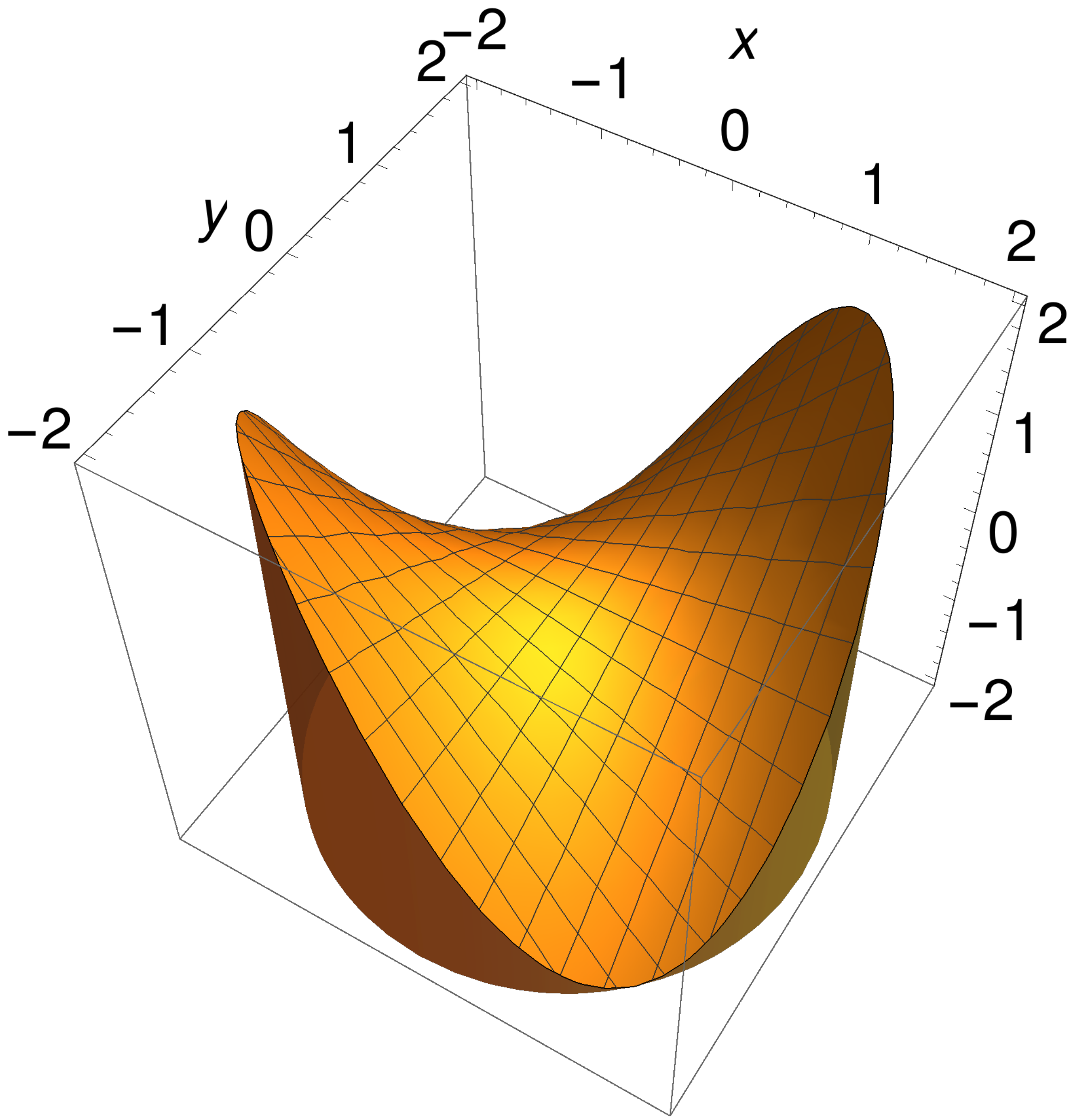
This also works for functions with more than two variables.

An example from before

Determines if $f(x, y) = 2\,x\,y$ have maximal and minimal values for

$$(x, y) \in D = \left\{(x, y) : x^2 + y^2 \leq 4\right\}$$

and determine where they are.



An example from before — with new method

Finding the extreme points on the boundary is equivalent to finding the extreme values of $f(x, y) = 2\,x\,y$, restricted to the curve $g(x, y) = x^2 + y^2 - 4 = 0$.

The corresponding Lagrange function is

$$L(x,\ y,\ \lambda) = \lambda\left(x^2 + y^2 - 4\right) + 2\,x\,y$$

To find the critical points, we need to solve $\nabla L(x, y, \lambda) = 0$,

$$2\,\lambda\,x + 2\,y = 0$$
$$2\,x + 2\,\lambda\,y = 0$$
$$x^2 + y^2 - 4 = 0$$

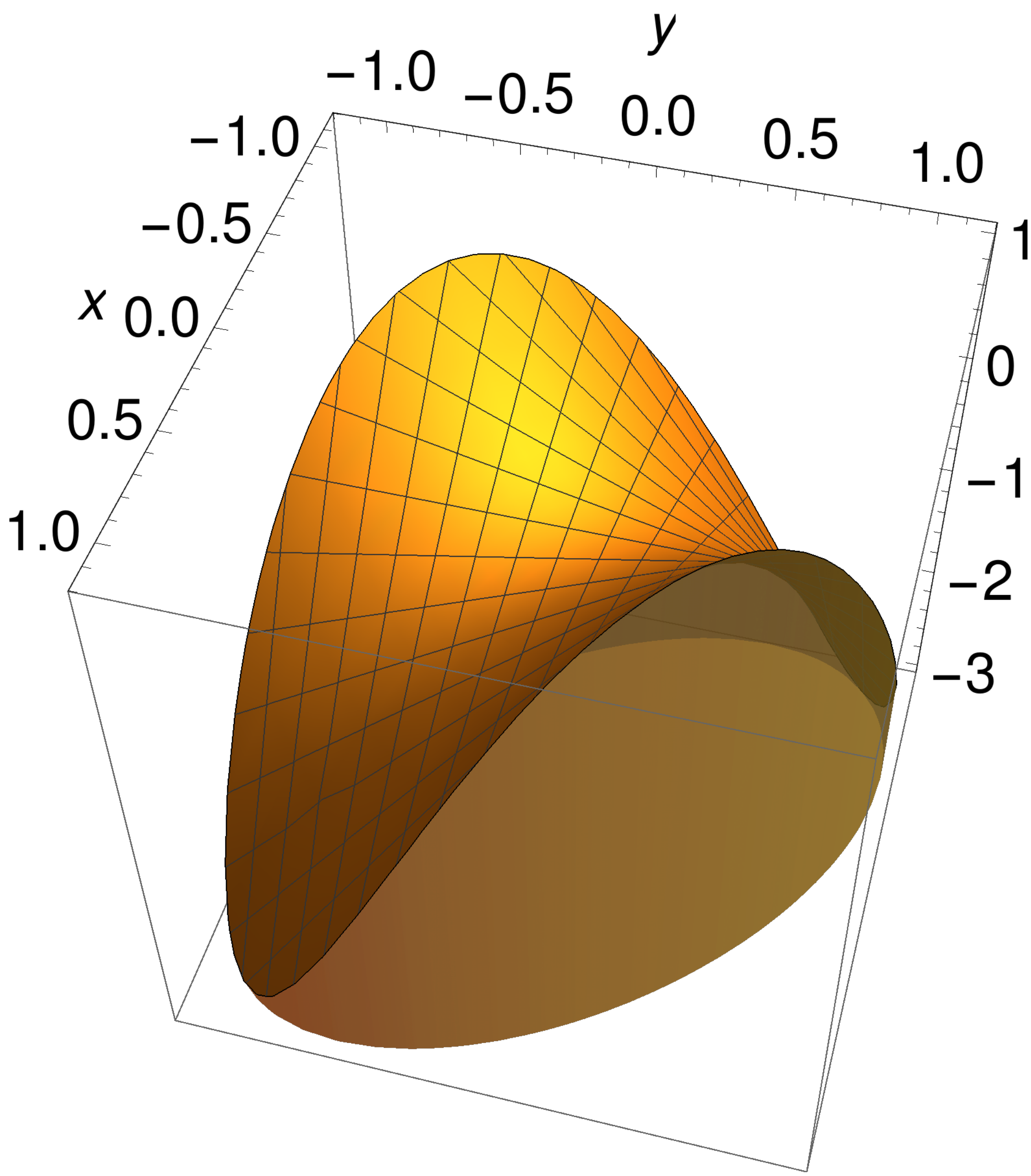
The solutions and the corresponding $f(x, y)$ are

x	y	λ	f(x,y)
$-\sqrt{2}$	$-\sqrt{2}$	-1	4
$-\sqrt{2}$	$\sqrt{2}$	1	-4
$\sqrt{2}$	$-\sqrt{2}$	1	-4
$\sqrt{2}$	$\sqrt{2}$	-1	4

So the function has two maximal points $\left(\pm\sqrt{2}, \pm\sqrt{2}\right)$ and two minimal points $\left(\mp\sqrt{2}, \pm\sqrt{2}\right)$.

Possible exam problem !!

Determine the maximal and minimal value of the function $f(x, y) = 3\,x\,y$ in the domain given by the inequality $x^2 + x\,y + y^2 \leq 1$.



Solution

Critical points: $\nabla f = (3y, 3x)$ and the only critical the point is (0, 0) and it's a saddle point.

Boundary points: we use the Lagrange multipliers method. We define

$$L(x, y, \lambda) = 3xy + \lambda\left(x^2 + xy + y^2 - 1\right)$$

To find the critical points of L, we need to solve $\nabla L(x, y, \lambda) = 0$, i.e.,

$$\lambda(2x + y) + 3y = 0$$
$$\lambda(x + 2y) + 3x = 0$$
$$x^2 + xy + y^2 - 1 = 0$$

The solutions and the corresponding $f(x, y)$ are

x	y	λ	f(x,y)
-1	1	3	-3
1	-1	3	-3
$-\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	-1	1
$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	-1	1

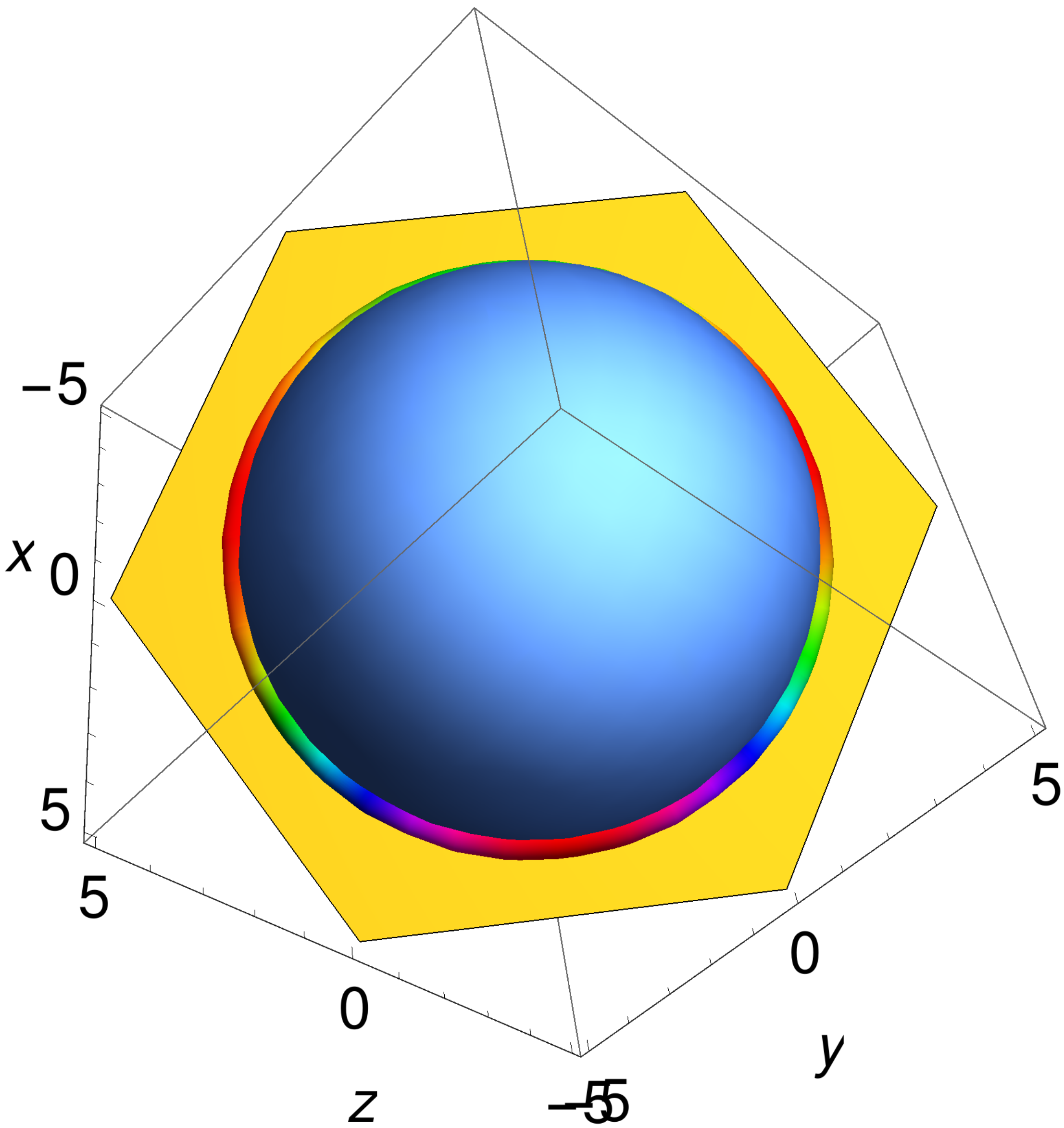
Thus the function has minimal value −3 and maximal value 1.

Lagrange's multiplier method for more than one constraints

To find extreme values of $f(x, y, z)$ under the constraints $g(x, y, z) = 0$ and $h(x, y, z) = 0$, we can similarly look for critical points the Lagrange function

$$L(x, y, z, \lambda, \mu) = f(x, y, z) + \lambda\, g(x, y, z) + \mu\, h(x, y, z)$$

Example Determine the maximal and minimal values of the function $f(x, y, z) = 2x + yz$ at the intersection of the plane $g(x, y, z) = x + y + z = 0$ and the sphere $h(x, y, z,) = x^2 + y^2 + z^2 - 24$.



Solution

The corresponding Lagrange function is

$$L(x, y, z, \lambda, \mu) = 2x + yz + \lambda(x + y + z) + \mu(x^2 + y^2 + z^2 - 24)$$

To find the critical points of L, we have to solve $\nabla L(x, y, z, \lambda, \mu) = 0$,

$$\lambda + 2\mu x + 2 = 0$$
$$\lambda + 2\mu y + z = 0$$
$$\lambda + y + 2\mu z = 0$$
$$x + y + z = 0$$
$$x^2 + y^2 + z^2 = 24$$

The solutions of these equations and the corresponding $f(x, y, z)$ are

x	y	z	λ	μ	f(x,y,z)
-4	2	2	-2	0	-4
-1	$\frac{1}{2}(1 - 3\sqrt{5})$	$\frac{1}{2}(1 + 3\sqrt{5})$	-1	$\frac{1}{2}$	-13
-1	$\frac{1}{2}(1 + 3\sqrt{5})$	$\frac{1}{2}(1 - 3\sqrt{5})$	-1	$\frac{1}{2}$	-13
4	-2	-2	$\frac{2}{3}$	$-\frac{1}{3}$	12

So $f(x, y, z)$ has minimal value −13 and maximal value 12 on the curve.

A building like a box

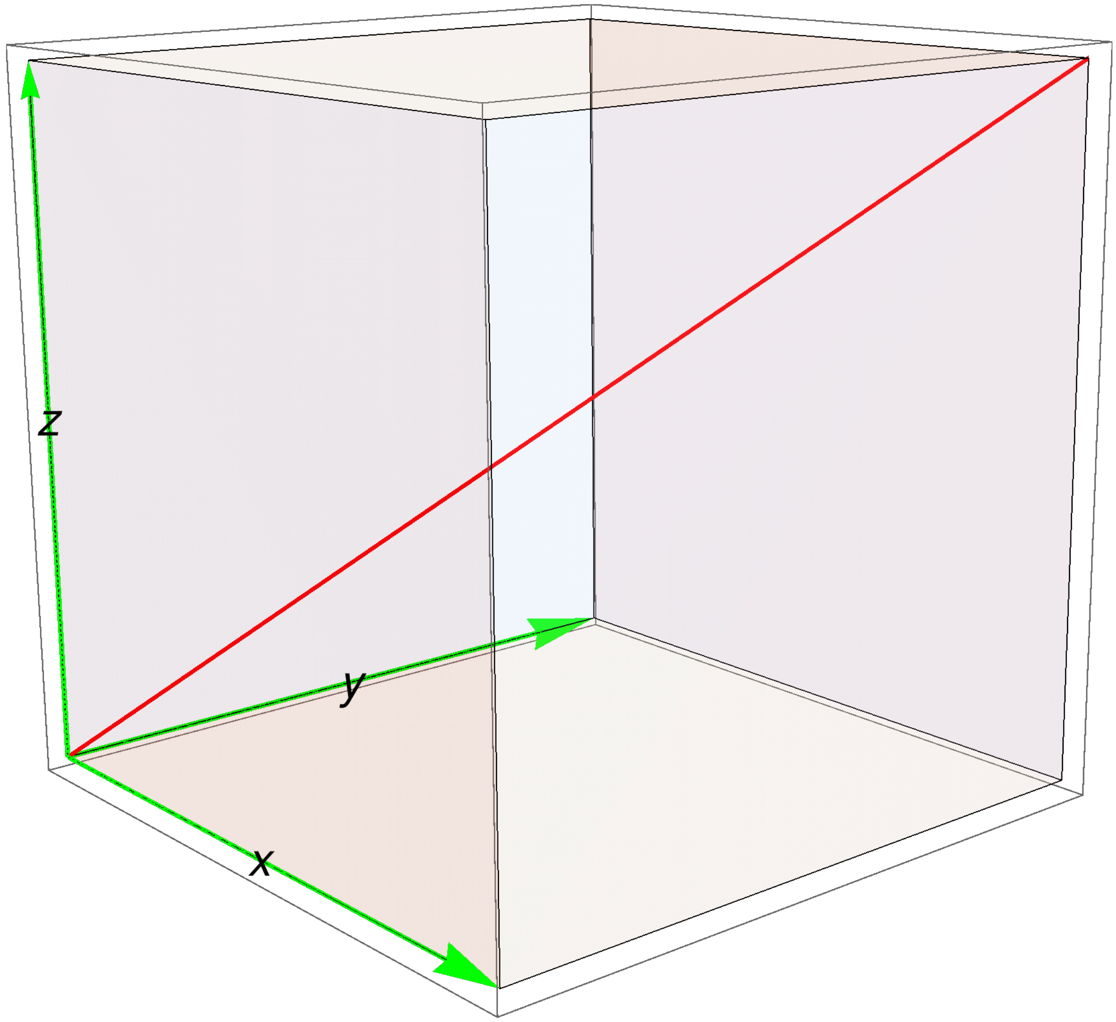
You are asked to design a building that looks like a box with the condition that the diagonal of the building must be 1 km. 

How can you maximize the volume of the building?

Quiz

Suppose the length of a box has diagonal is 1 meter. What's the maximum volume this box can have?

Solution This is equivalent to ask what is the maximum of $f(x, y, z) = x\,y\,z$ with the constraint that $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$ and $x, y, z \geq 0$.



Find the critical points for the Lagrange function

$$L(x, y, z, \lambda) = f(x, y, z) + \lambda\, g(x, y, z)$$

The critical points of $L(x, y, z, \lambda)$ must satisfy

$$2\lambda x + yz = 0$$
$$xz + 2\lambda y = 0$$
$$xy + 2\lambda z = 0$$
$$x^2 + y^2 + z^2 - 1 = 0$$
$$x > 0$$
$$y > 0$$
$$z > 0$$

The maximum point of $f(x, y, z)$ must be one of them.

Possible exam problem

Find the maximal and minimal value of the function $f(x, y, z) = x\,y + y\,z + z\,x$ in the region D in which $x^2 + y^2 + z^2 \le 1$.

Hint:

- Find the critical points of f in D and classify them into max./min./saddle points
- Use Lagrange multiplier method to find the extreme values of $f(x, y, z)$ on the curve $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$.
- Compare all the local max/min to find the absolute max and min.