Lecture 13
14.1 Double integral
14.2 Iterated integration

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Several Variable Calculus, 1MA017, Autumn 2019

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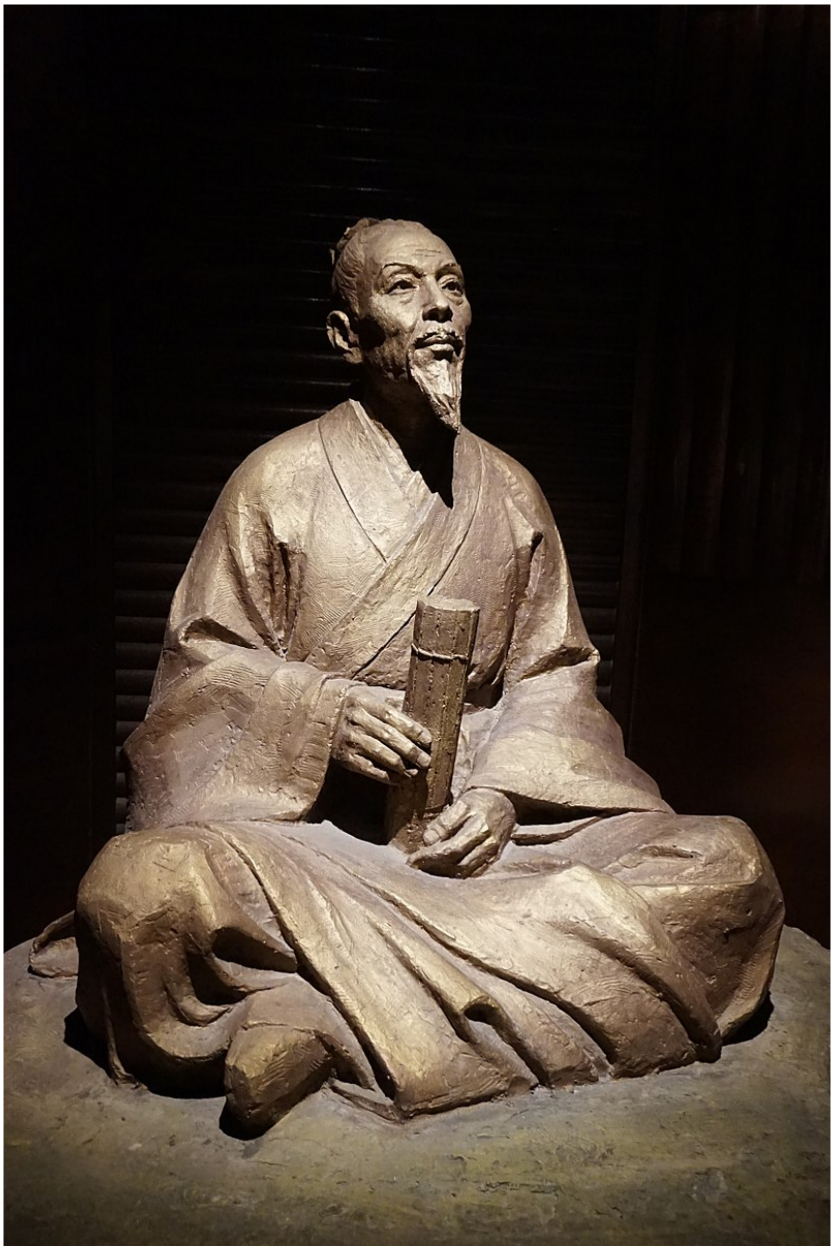
What is π

The area of a circle of radius r is π

$\pi = 3.1415926535897932385 ...$



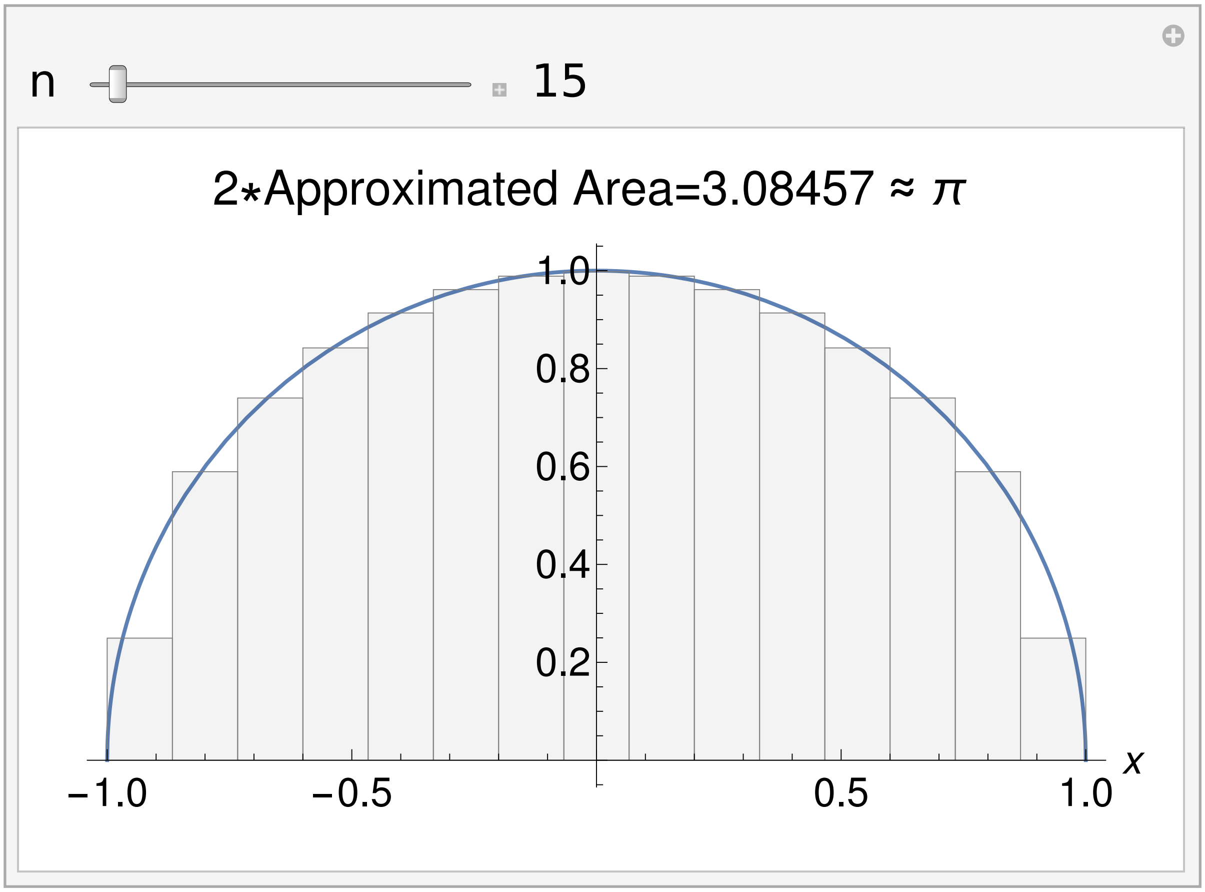
Zu Chongzhi (429–500 AD), Chinese mathematician. He calculated π as between 3.1415926 and 3.1415927, a record which held for **800** years.

One way to compute π

The area under the curve $f(x) = \sqrt{1 - x^2}$ between −1 and 1 is $\frac{\pi}{2}$.

We can approximate it with boxes under the curve.

The narrower the boxes we use, the better the approximation we have.



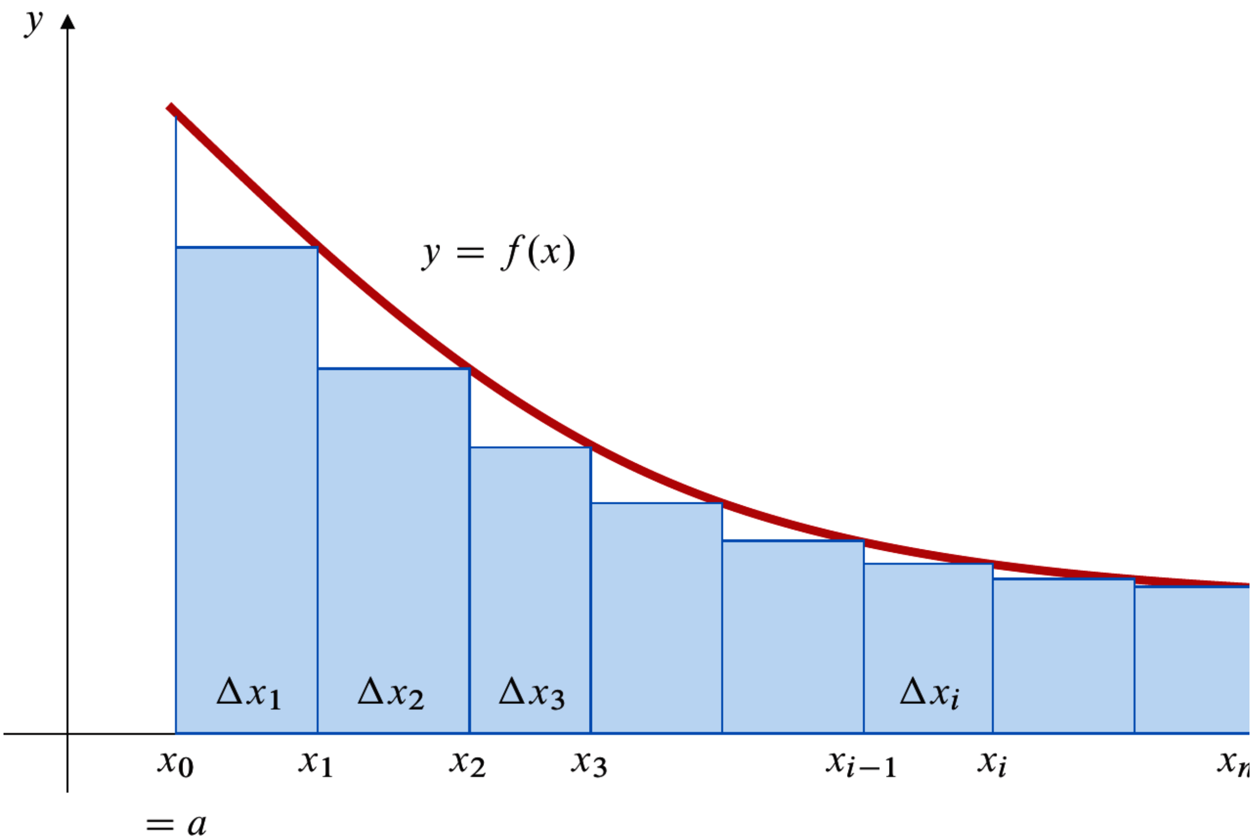
What is an integral?

We can approximate the area below a curve $f(x)$ between a and b by the Riemann sum

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

where $x_{i-1} \leq x_i^* \leq x_i$.

This becomes a better and better approximation when $\max(\Delta x_i)$ decreases.



What is an integral?

We define the integral of $f(x)$ over the interval $[a, b]$ as

$$\int_a^b f(x)\, dx = \lim \sum_{i=1}^{n} f(x_i^*)\, \Delta x_i$$

where the limit is taken as $\max(\Delta x_i) \to 0$ and $n \to \infty$.

This can be seen as the area under $f(x)$ between $x = a$ and $x = b$.

The fundamental theorem of calculus says if $F'(x) = f(x)$, then

$$\int_a^b f(x)\, dx = F(a) - F(b)$$

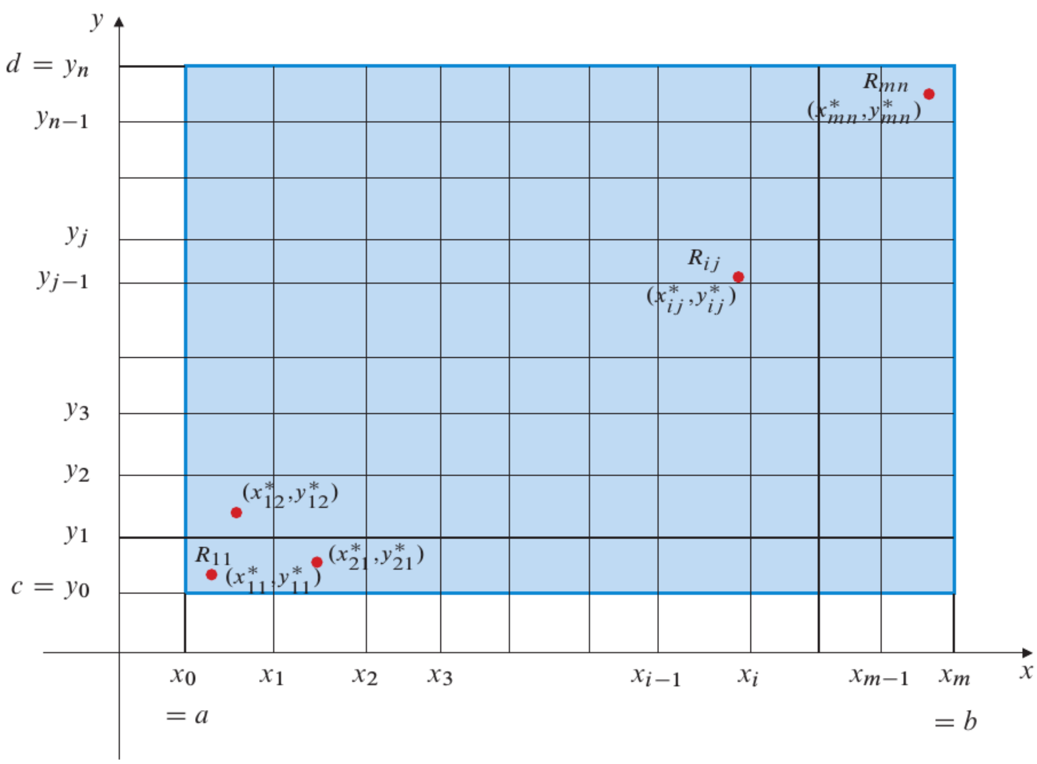
The volume under a surface

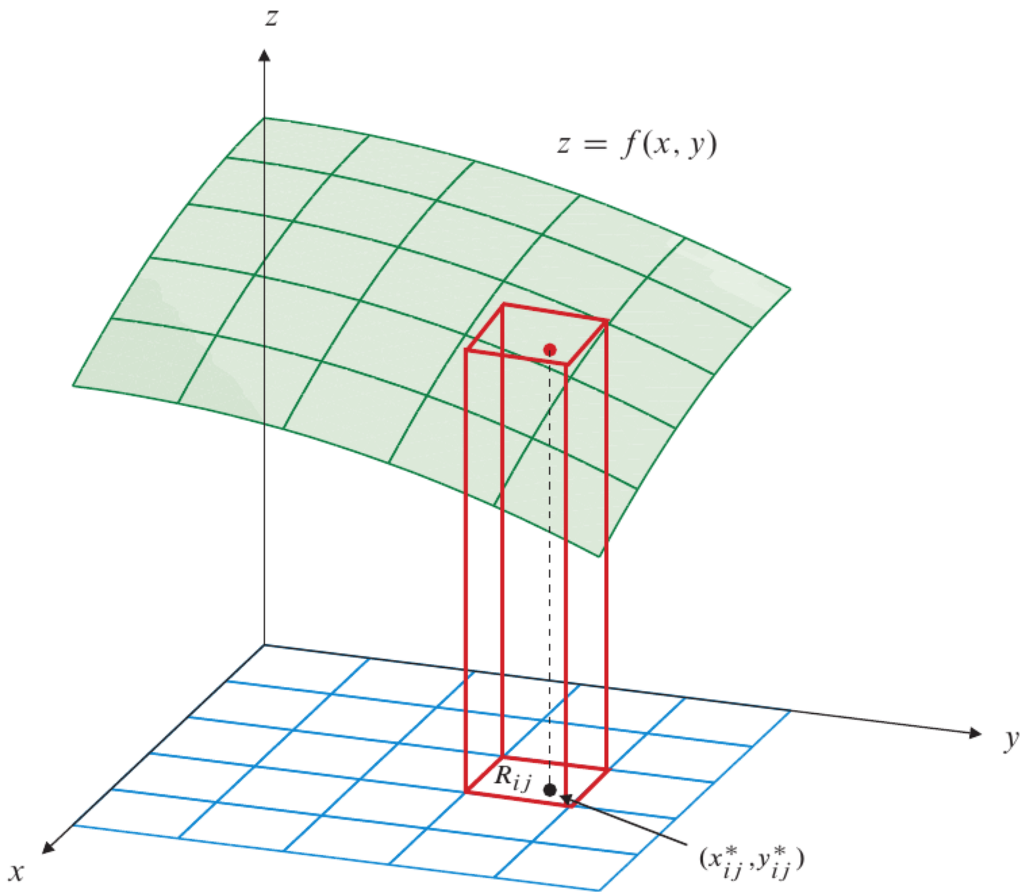
The volume under a surface $f(x, y)$ within a rectangle

$D = \{(x, y) : a \le x \le b, c \le y \le d\}$

can be approximated similarly by the Riemann sum

$$\sum_{i=1}^{n} \sum_{j=1}^{n} f(x^*_{i,j}, y^*_{i,j})\, \Delta A_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{i,j}, y^*_{i,j})\, \Delta x_i\, \Delta y_j$$




z
z = f(x, y)
y
x
R_ij
(x*_ij, y*_ij)

What is a double integral?

We define the double integral of a bounded function $f(x, y)$ in the rectangle D

$$\iint_D f(x, y)\, dA = \lim \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{i,j}^{*}, y_{i,j}^{*}\right) \Delta A_{i,j}$$

where the limit is taken when $\max(\Delta x_i, \Delta y_i) \to 0$.

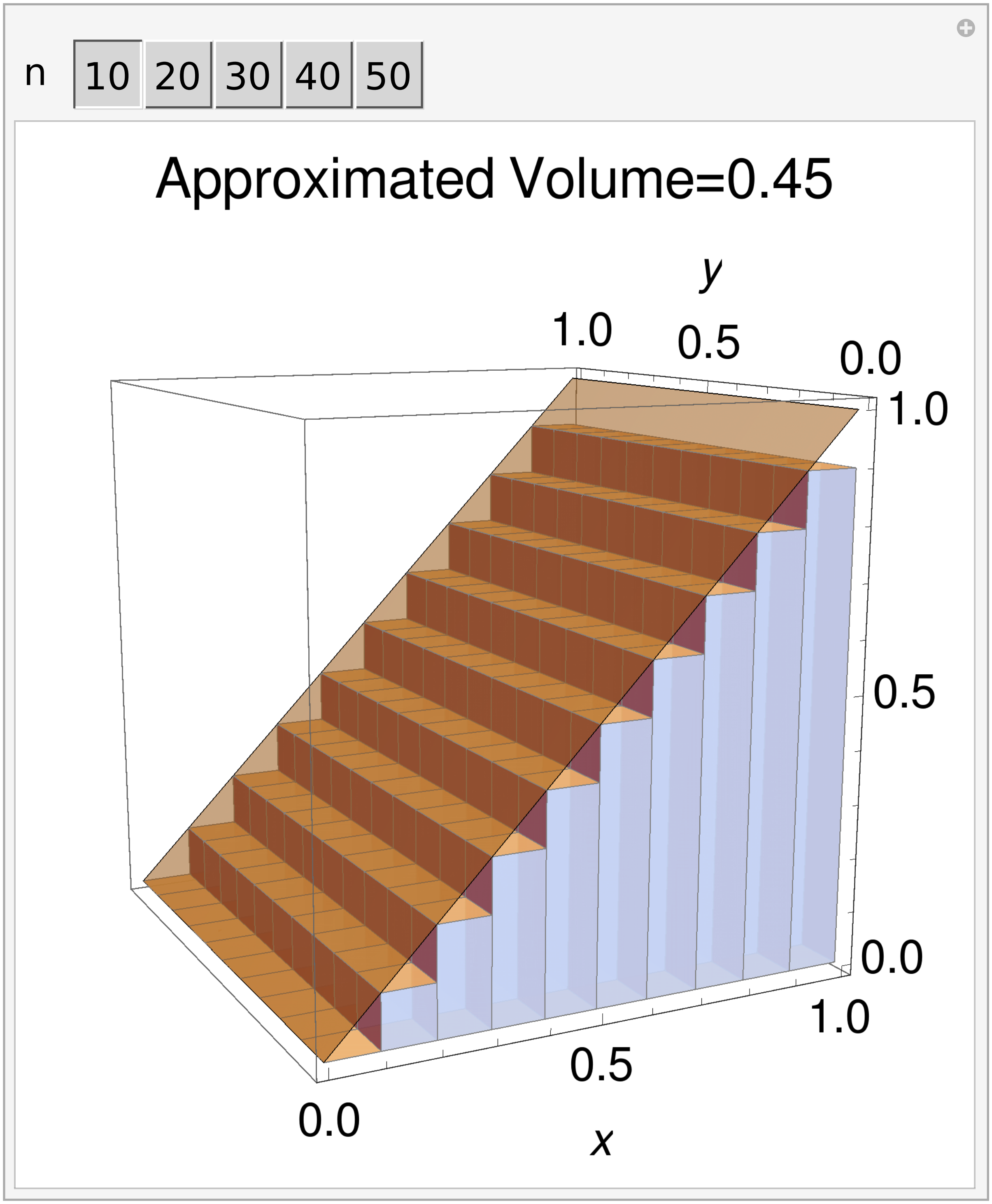
A function is integrable if this limit exist regardless of how we divide D into boxes.

This can be seen as the volume under the surface $f(x, y)$ within D.

Example

Let $f(x, y) = x$ and let $D = \{(x, y) : 0 \leq 1 \leq x,\ 0 \leq y \leq 1\}$. What is

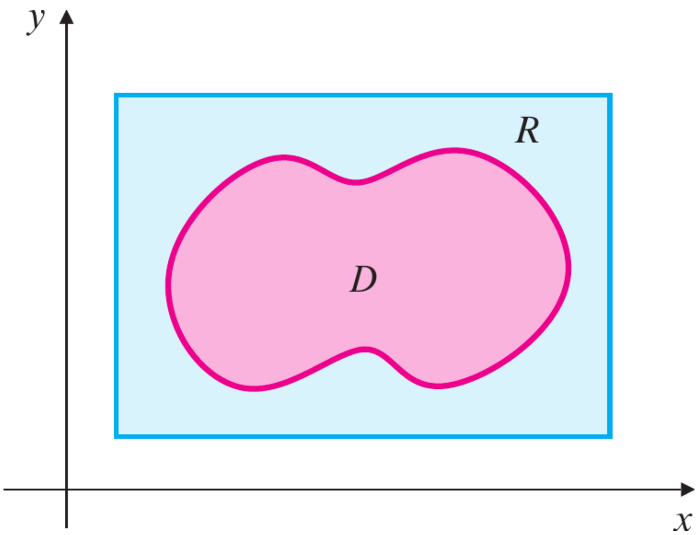
$$\iint_D f(x, y)\, dA$$


n
10
20
30
40
50
Approximated Volume=0.45
y
1.0
0.5
0.0
1.0
0.5
0.0
1.0
0.5
0.0
x

Double integral over more general domains

If $f(x, y)$ is defined on a bounded domain D and R is a rectangle containing R, then we extend $f(x, y)$ to R be defining

$$\hat{f}(x, y) = \begin{cases} f(x, y) & \text{if } (x,y) \in \\ 0 & \text{Otherwise} \end{cases}$$



Then we define

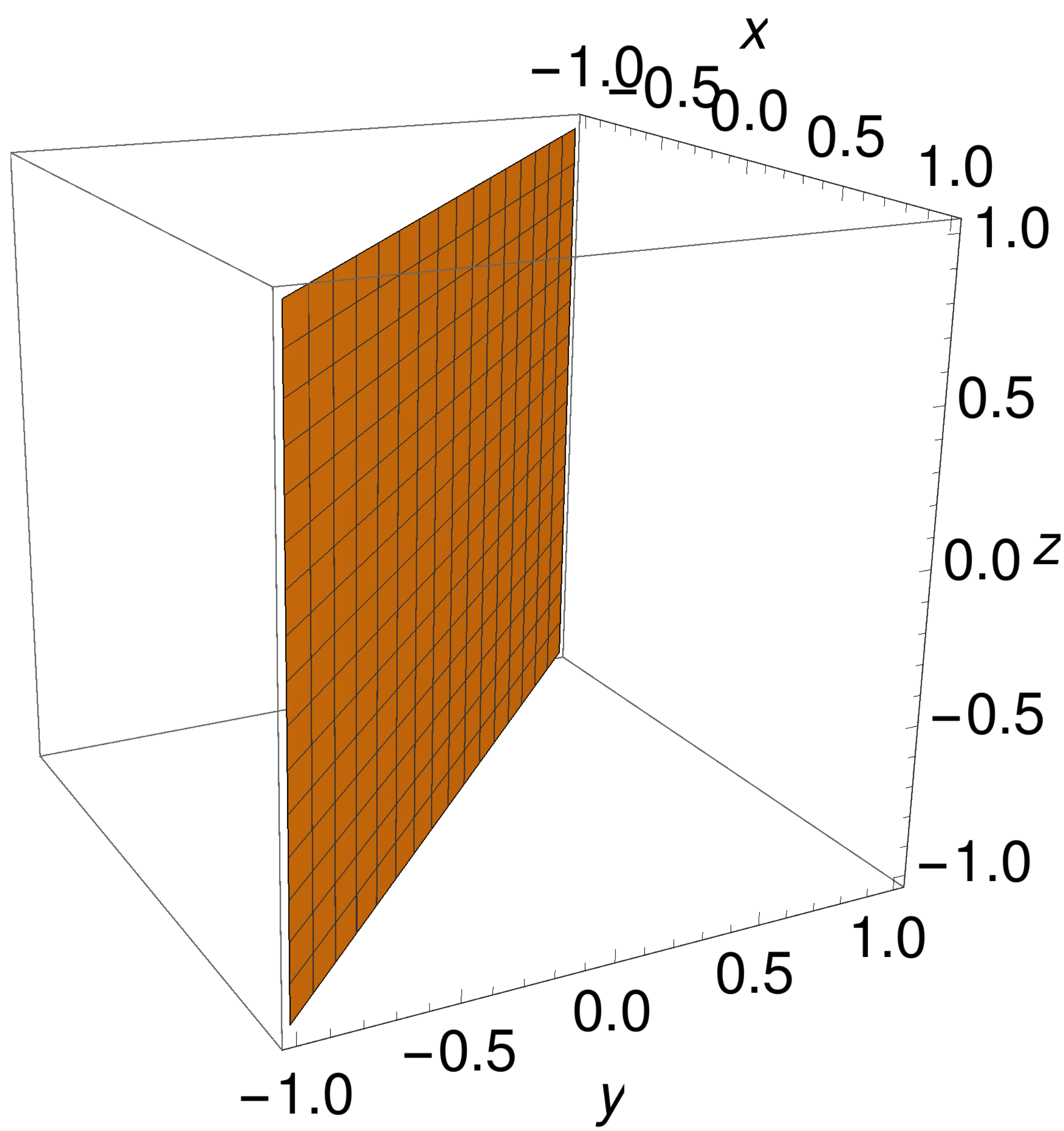
$$\iint_D f(x, y)\, dA = \iint_R \hat{f}(x, y)\, dA$$

Theorem

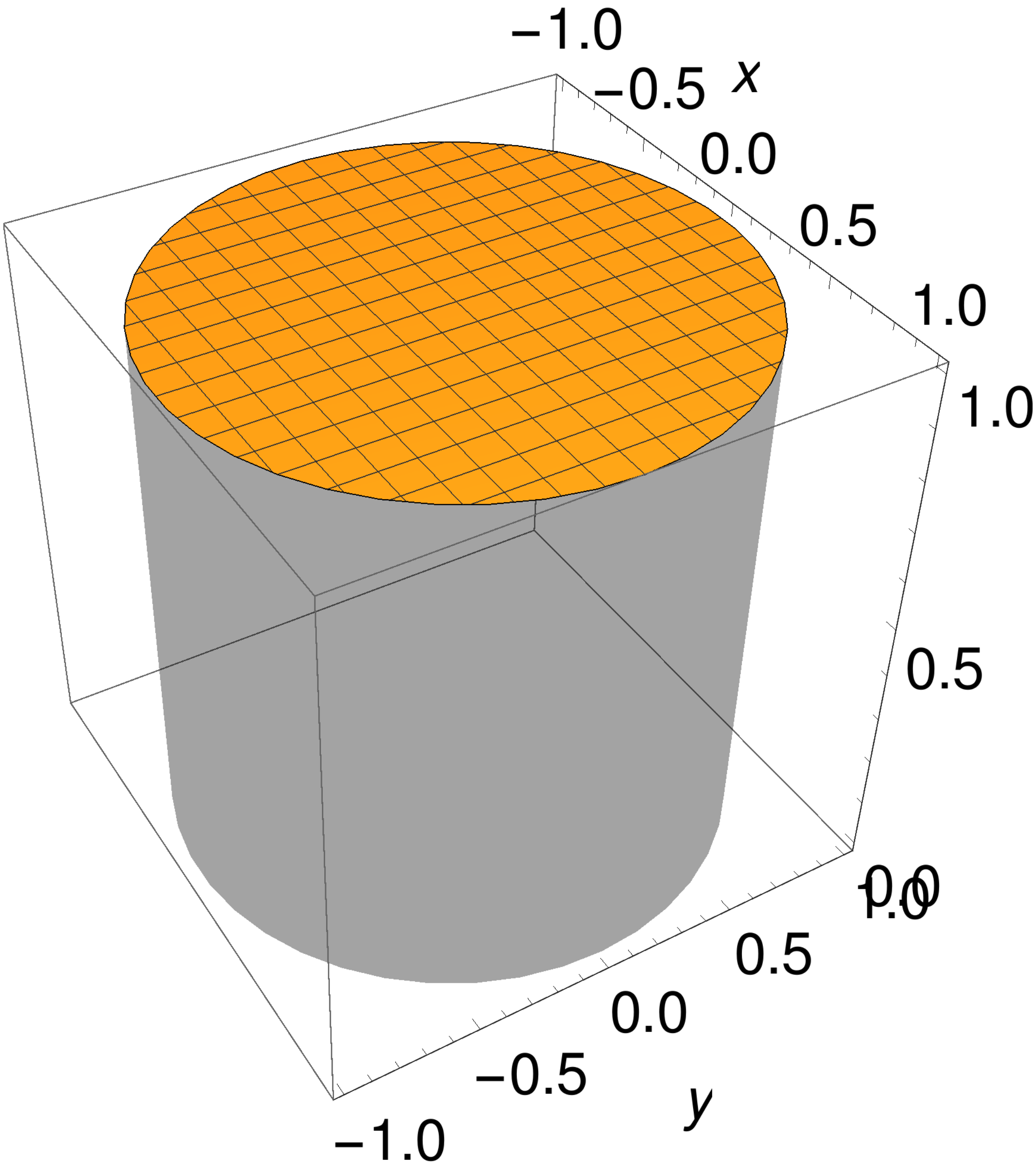
If f is **continuous** on a **closed**, **bounded** domain D whose boundary consists of finitely many curves of finite length, then f is integrable on D.

Properties of double integral

- $\int\int_D f(x, y)\, dA = 0$ if D has zero area.

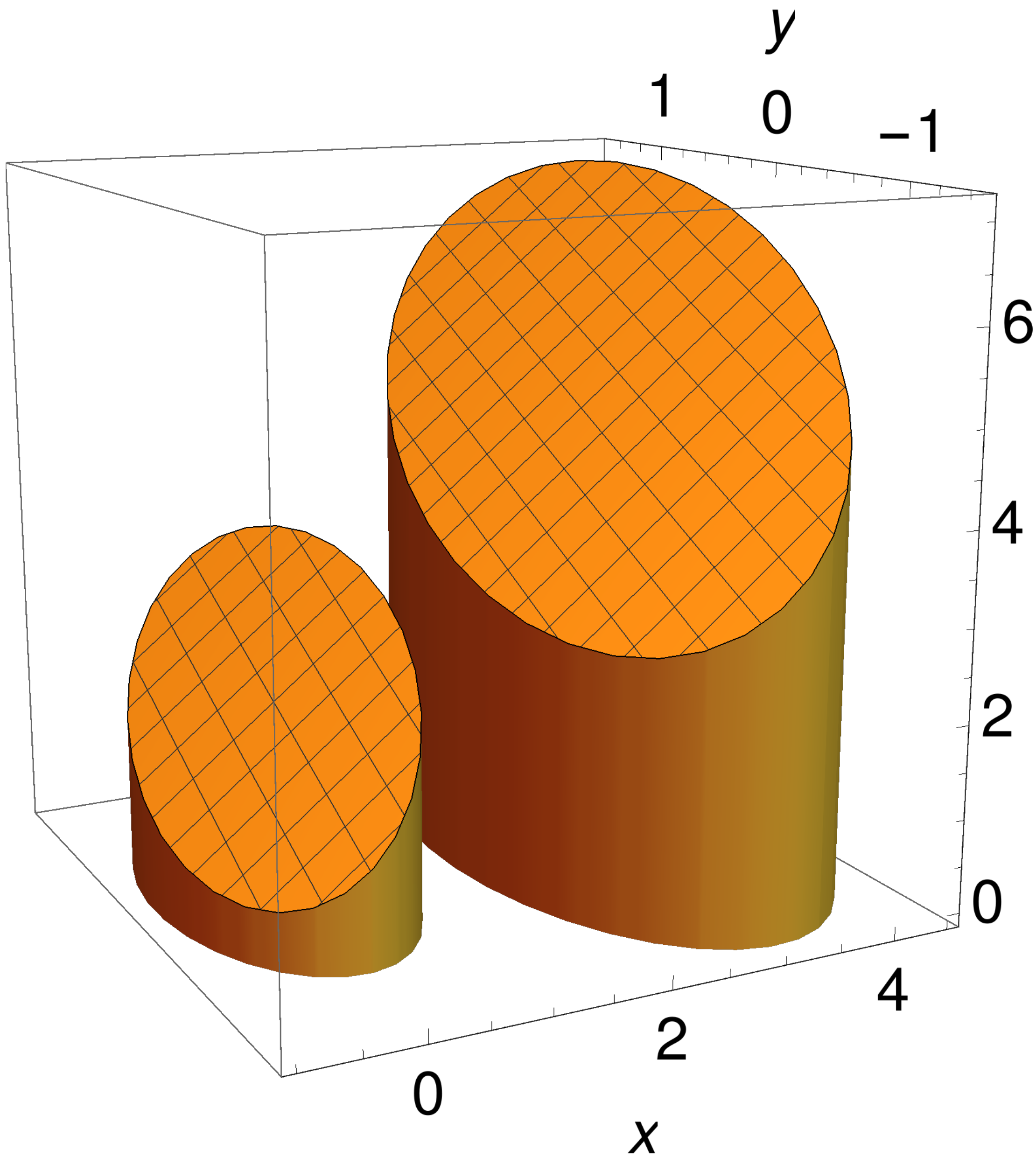


- $\int\int_D 1\, dA =$ area of D



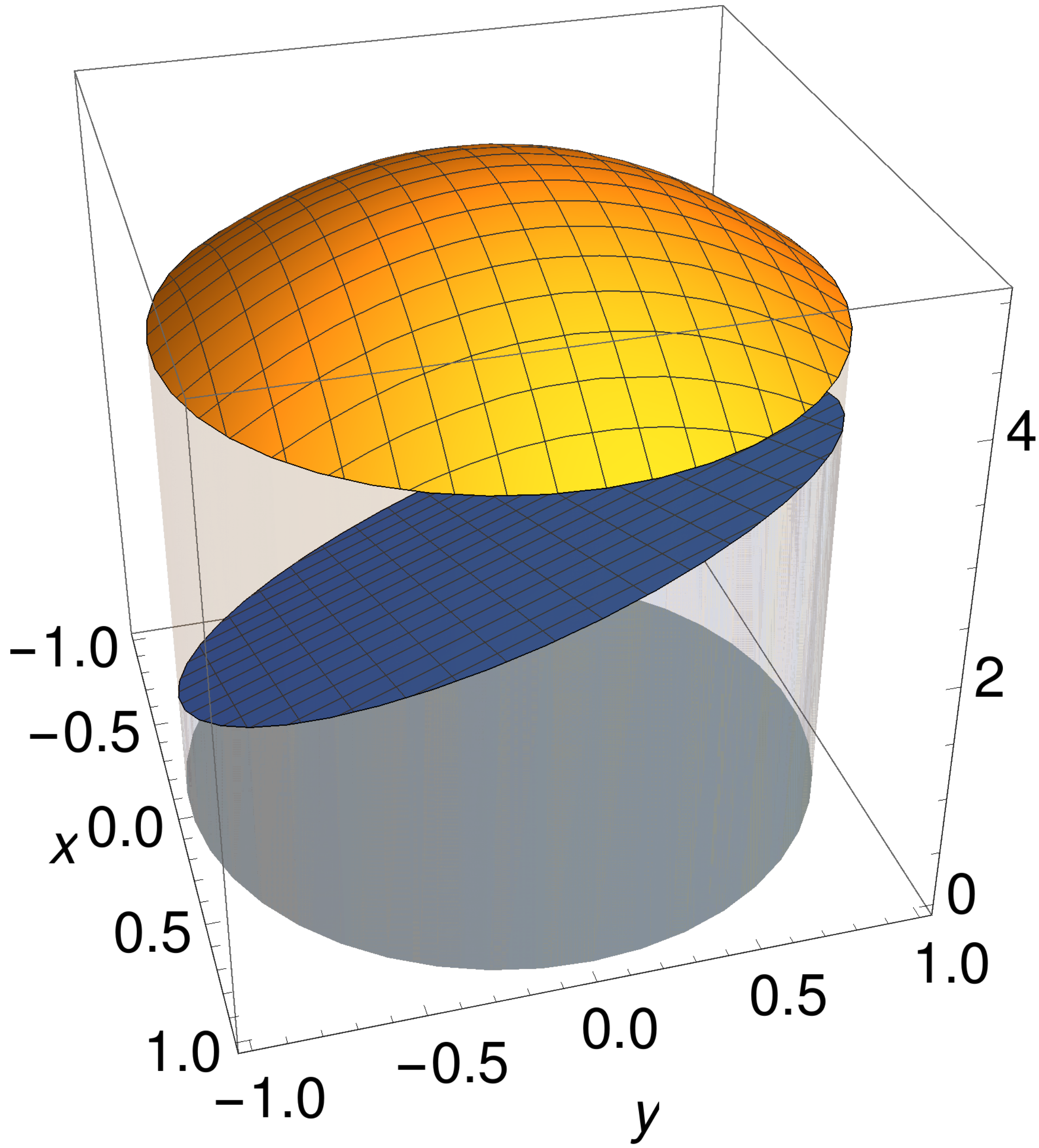
- The triangle inequality $\left|\int\int_D f(x, y)\, dA\right| \leq \int\int_D |f(x, y)|\, dA$
- If $D = D_1 \cup D_2$, then

$$\int\int_{D_1} f(x, y)\, dA + \int\int_{D_2} f(x, y)\, dA = \int\int_D f(x, y)\, dA$$



- If $f(x, y) \leq g(x, y)$, then

$$\iint_D f(x, y)\, dA \leq \iint_D g(x, y)\, dA$$

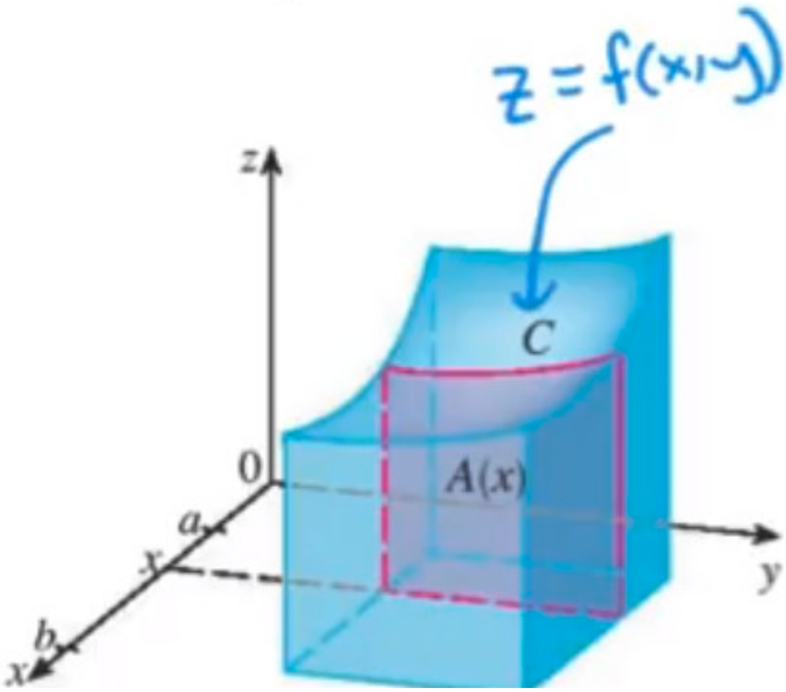

−1.0
−0.5
x 0.0
0.5
1.0
−1.0
−0.5
0.0
0.5
1.0
y
0
2
4

How to calculate double integral?

We can calculate a double integral by compute single integral twice. Let

$$A(x) = \int_c^d f(x, y)\, dy$$

which is the area of the slice at x.



Then

$$\iint_D f(x, y)\, dA = \int_a^b A(x)\, dx = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx$$

We can also integrate with respect to x first, i.e.,

$$\iint_D f(x, y)\, dA = \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy$$

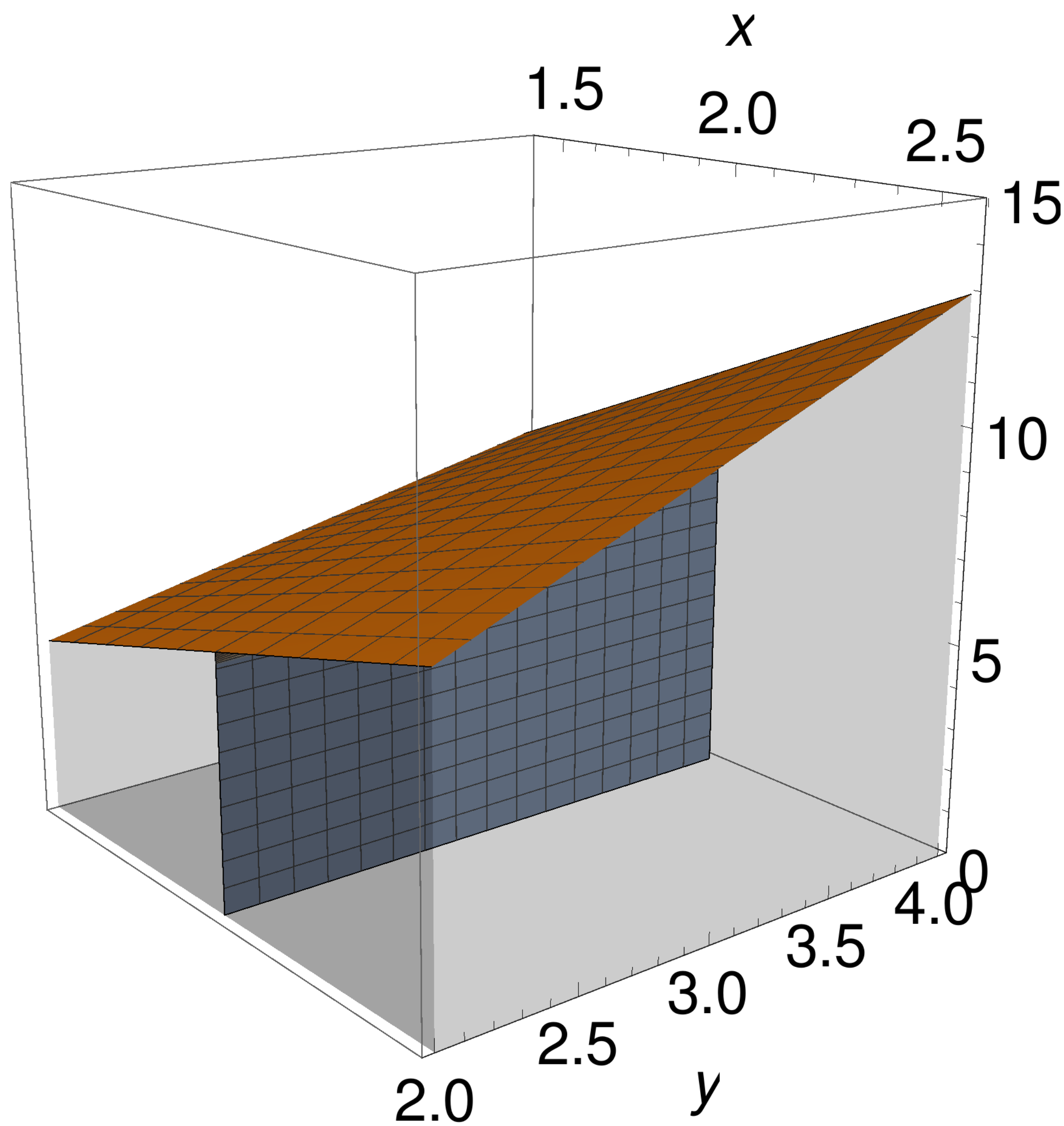
Example

If $D = \{(x, y) : 1 \le x \le 3,\ 2 \le y \le 4\}$, then

$$\iint_D (x + x\,y)\,dA = \int_1^3 \left(\int_2^4 (x + x\,y)\,dy\right) dx$$

$$= \int_1^3 \left(x\,y + \frac{x\,y^2}{2}\right)\Bigg|_2^4 dx$$

$$= \int_1^3 8\,x\,dx = 32$$



Quiz

Compute the double integral

$$\iint_D x\cos(y)\,dA$$

for $D = \{(x, y) : 2 \le x \le 8,\ 0 \le y \le \pi/2\}$.

Hint: Compute the following

$$\int_0^{\frac{\pi}{2}} \int_2^8 x\cos(y)\,dx\,dy$$

Which is the correct answer?

$$0,\ \frac{1}{2},\ 2,\ \frac{9}{2}$$

x-simple and *y*-simple area

A region D is y-simple if it can be described as

$D = \{(x, y) : a \leq x \leq b,\ c(x) \leq y \leq d(x)\}$

A region D is x-simple if it can be described as

$D = \{(x, y) : a(y) \leq x \leq b(y),\ c \leq y \leq d\}$

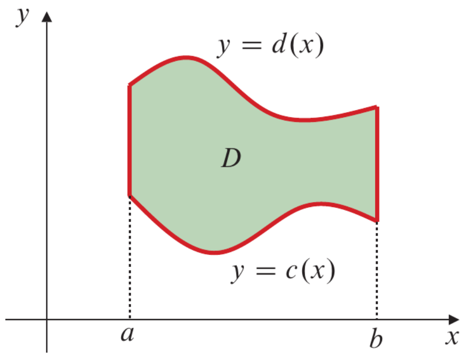


Figure 14.10 A y-simple domain

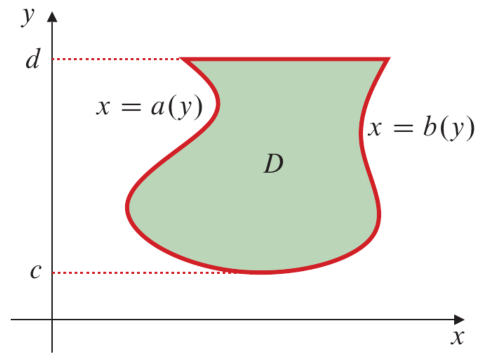
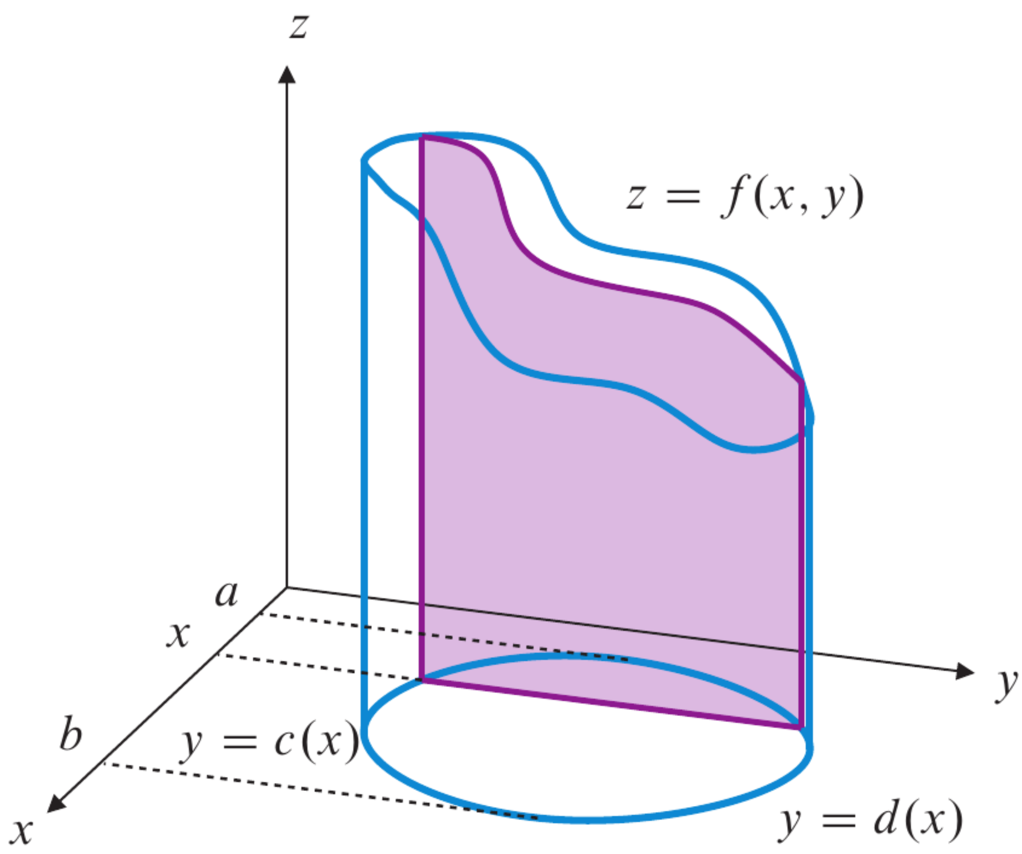


Figure 14.11 An x-simple domain

Integrate over *y*-simple area

If *D* is *x*-simple, then

$$\int\int_D f(x, y)\, dA = \int_a^b \left(\int_{c(x)}^{d(x)} f(x, y)\, dy\right) dx$$

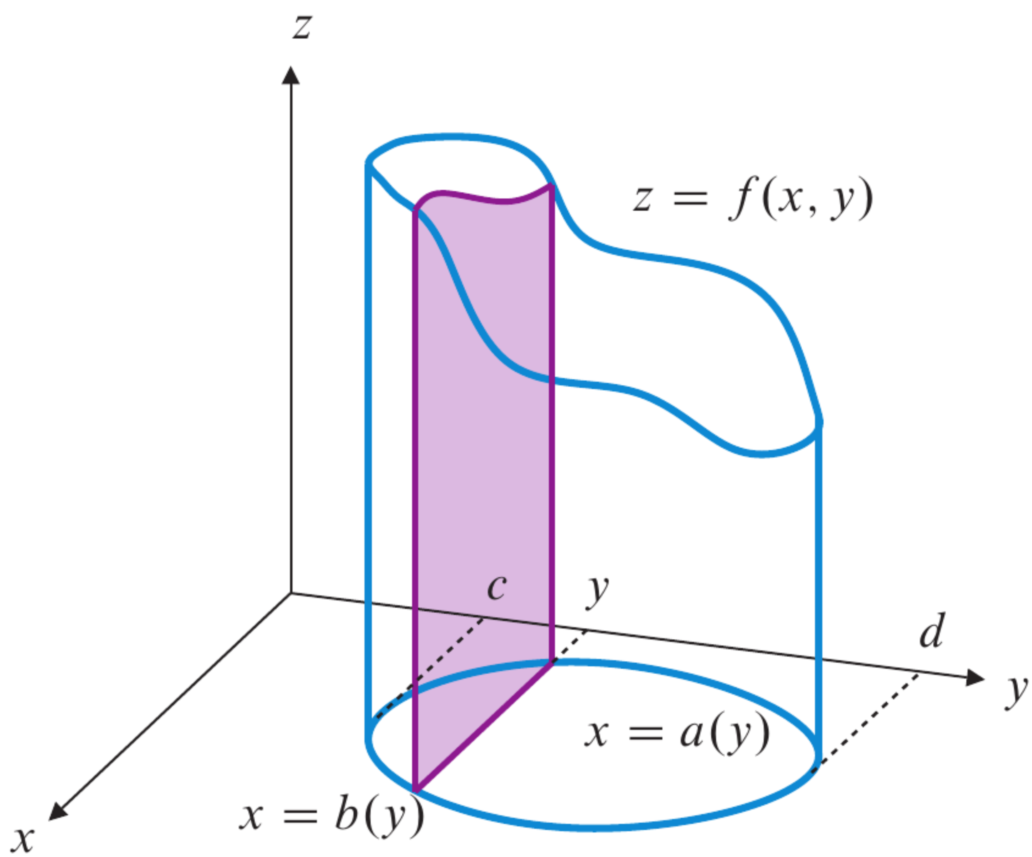


(a)

Integrate over *x*-simple area

If D is x-simple, then

$$\int\int_D f(x, y)\, dA = \int_c^d \left(\int_{a(y)}^{b(y)} f(x, y)\, dx\right) dy$$

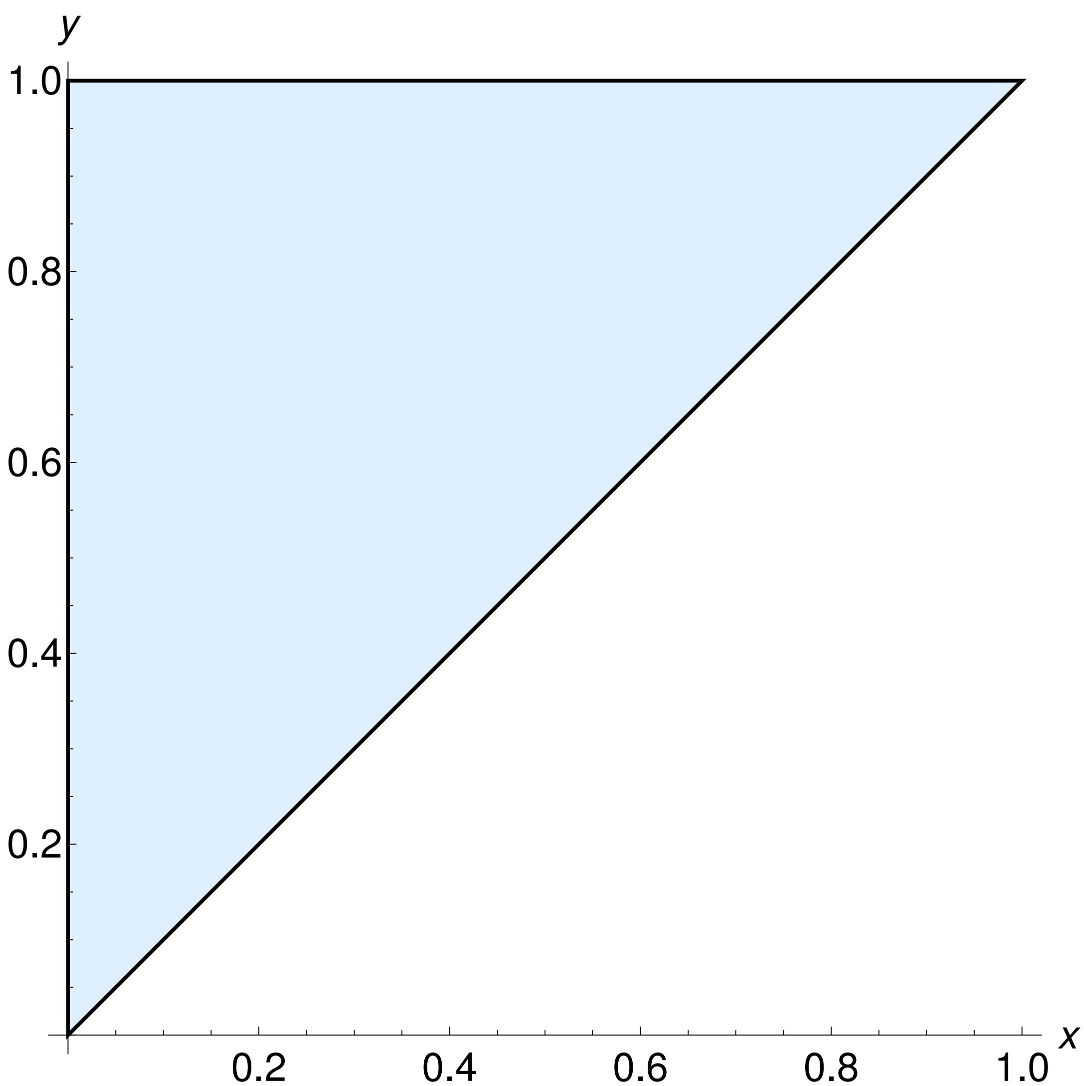


Example

Compute the integral

$$\iint_D x\, y\, dA$$

when *D* is the blue triangle in the picture.



Solution

$$\iint_D x\, y\, dA = \int_0^1 \int_x^1 x\, y\, dy\, dx$$

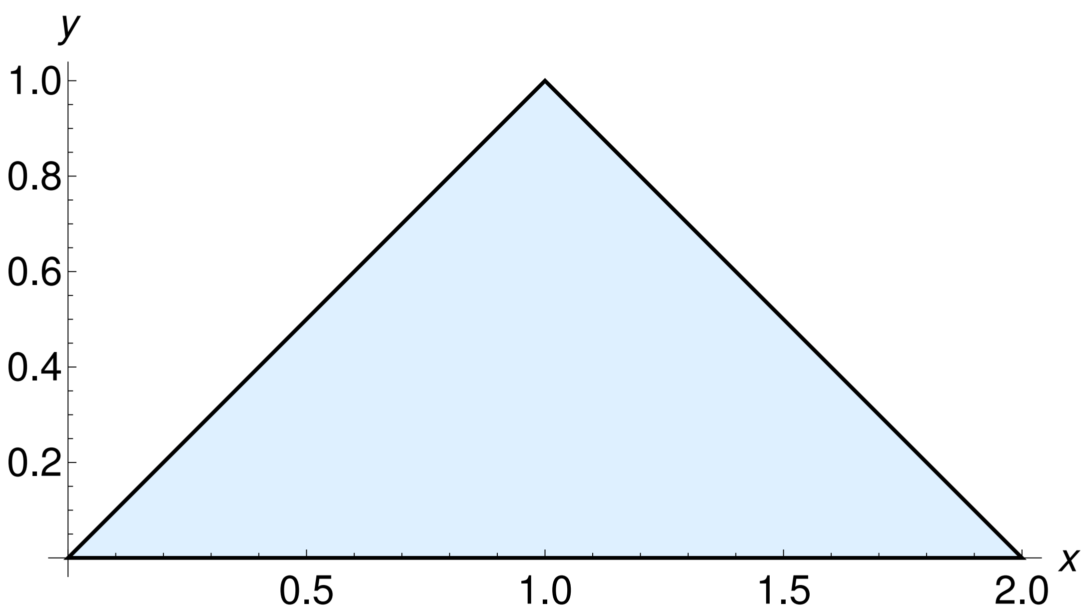
$$= \int_0^1 x \left(\frac{1}{2} - \frac{x^2}{2}\right) dx = \frac{1}{8}$$

Quiz

Compute the integral

$$\iint_D x\, y\, dA$$

when *D* is the blue triangle in the picture.



Hint: Compute

$$\iint_D x\, y\, dA = \int_0^1 \int_y^{2-y} x\, y\, dx\, dy$$

Which one is the correct answer?

$$-\frac{5}{3},\ -\frac{2}{3},\ \frac{1}{3},\ \frac{4}{3}$$