Lecture 14
14.2 Iterated integration
14.4 Integral in Polar Coordinates

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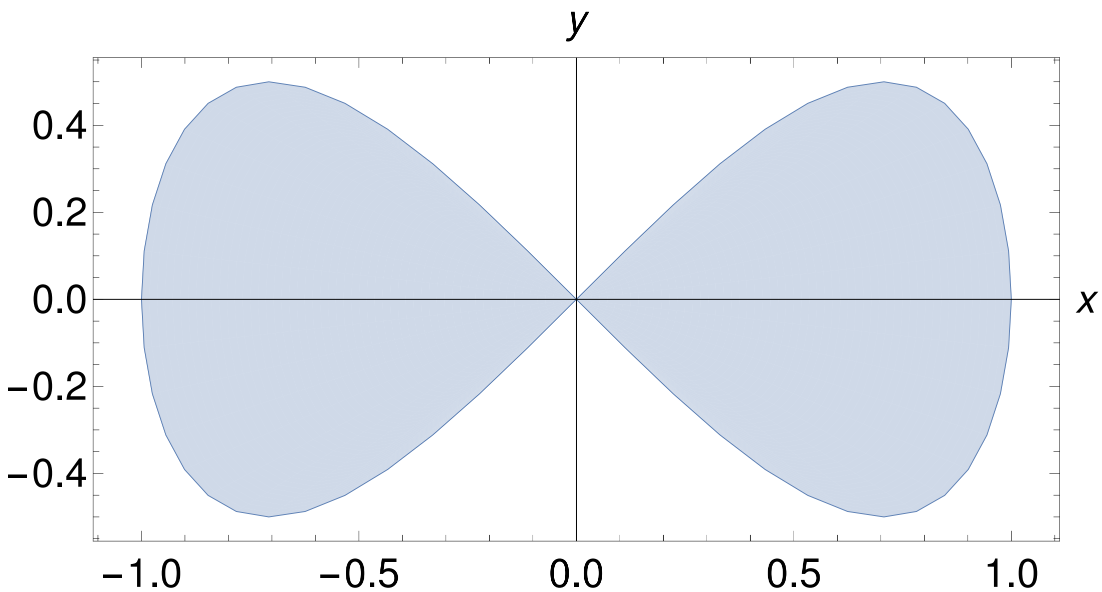
Symmetric area

An area $D \subset \mathbb{R}^2$ is symmetric over the x-axis if the following holds: If $(x, y) \in D$, then $(-x, y) \in D$.

An area $D \subset \mathbb{R}^2$ is symmetric over the y-axis if the following holds: If $(x, y) \in D$, then $(x, -y) \in D$.



Symmetry and integration

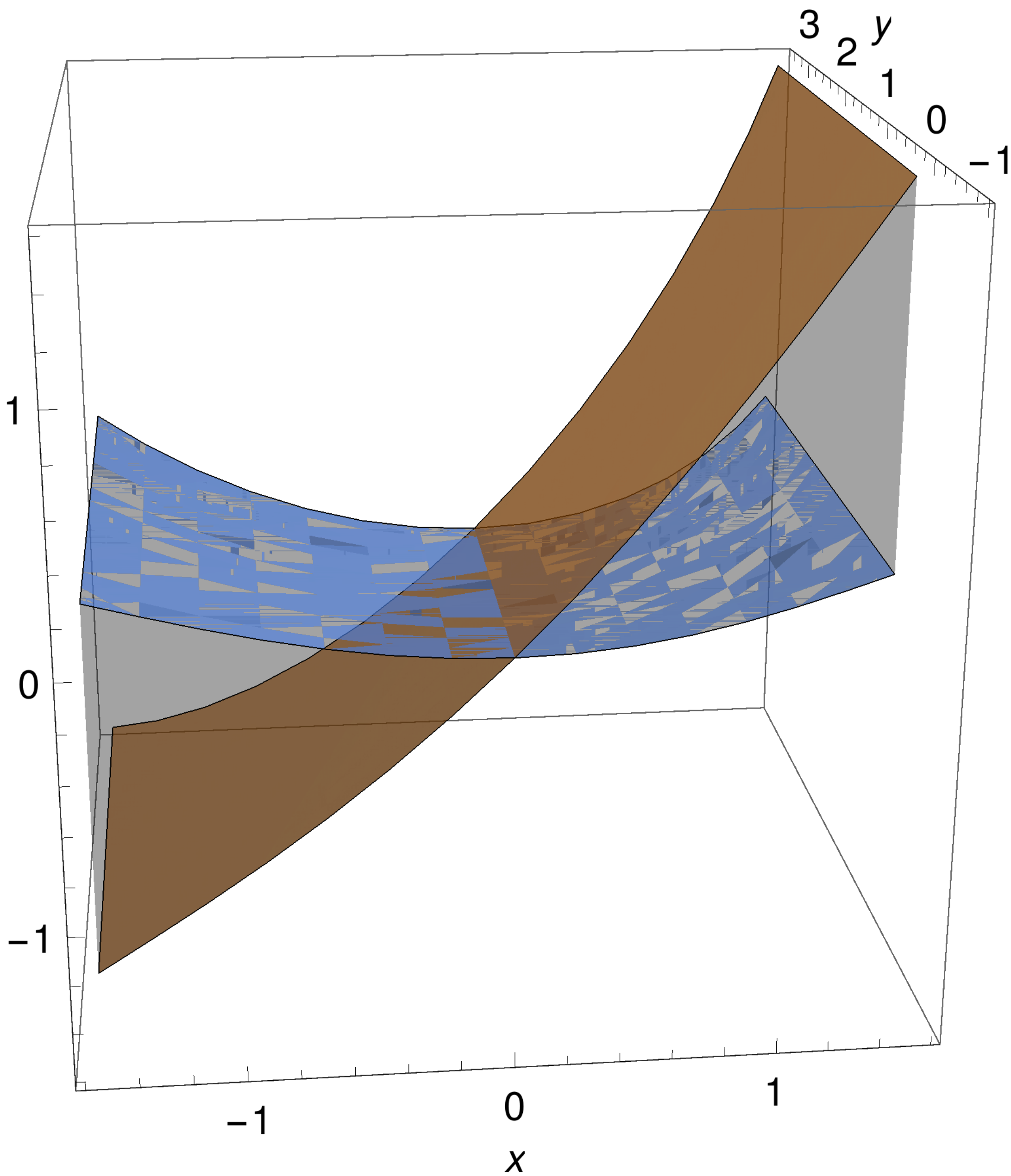
If $f(-x, y) = -f(x, y)$ and D is symmetric over y-axis, then

$$\iint_D f(x, y)\, dA = 0$$

Example

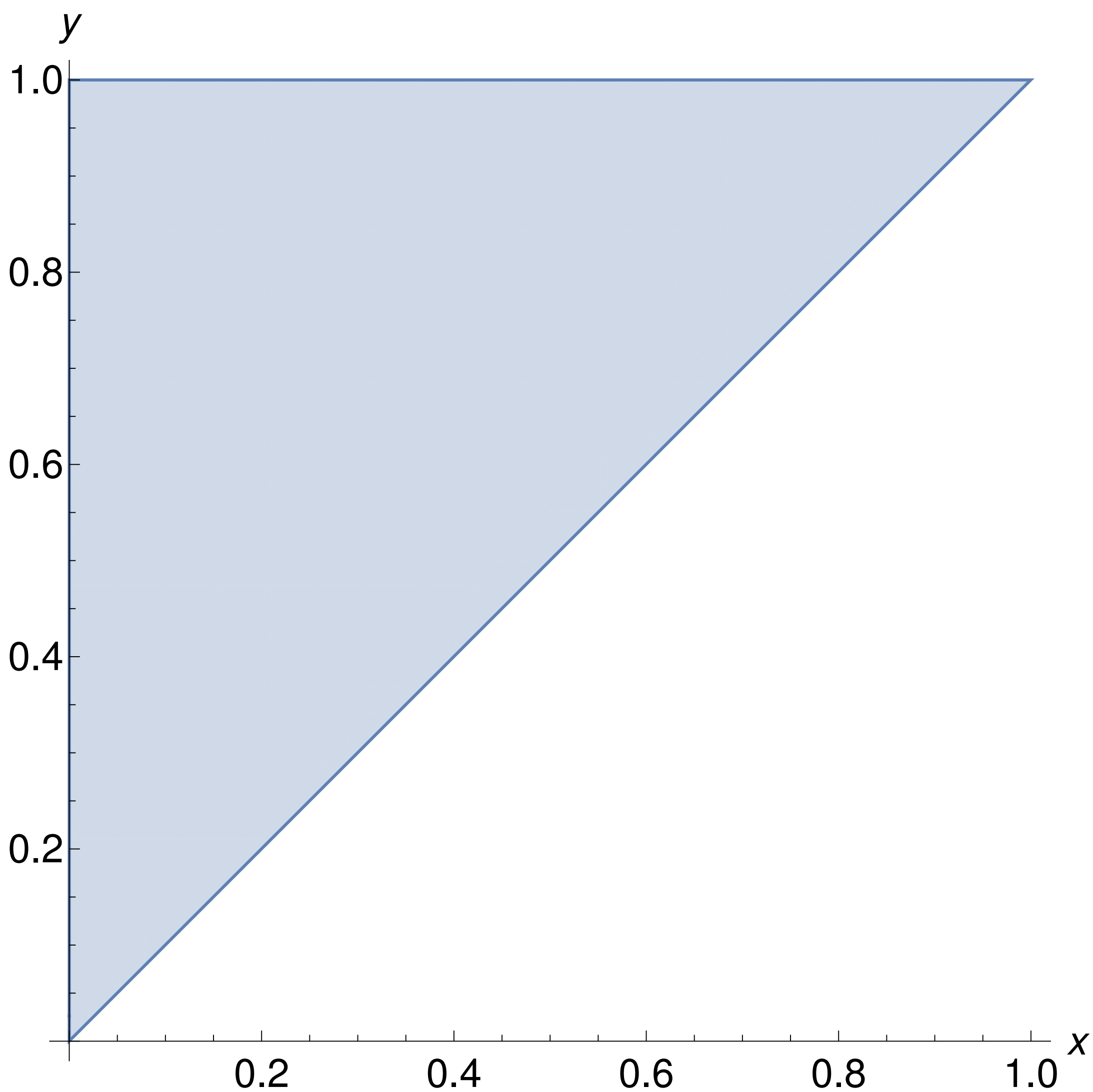
For $D = \left\{(x, y) : -\frac{\pi}{2} \le x \le \frac{\pi}{2}, -\cos(x) \le y \le 1 + x^2\right\}$, we have

$$\iint_D x\, dA = 0$$


3
2
y
1
0
−1
1
0
−1
−1
0
1
x

Quiz

Let D be as in the picture. Compute $\int\int_D e^{y^2}\, dA$.



It's seems difficult to compute

$$\int_0^1 \int_x^1 e^{y^2}\, dy\, dx$$

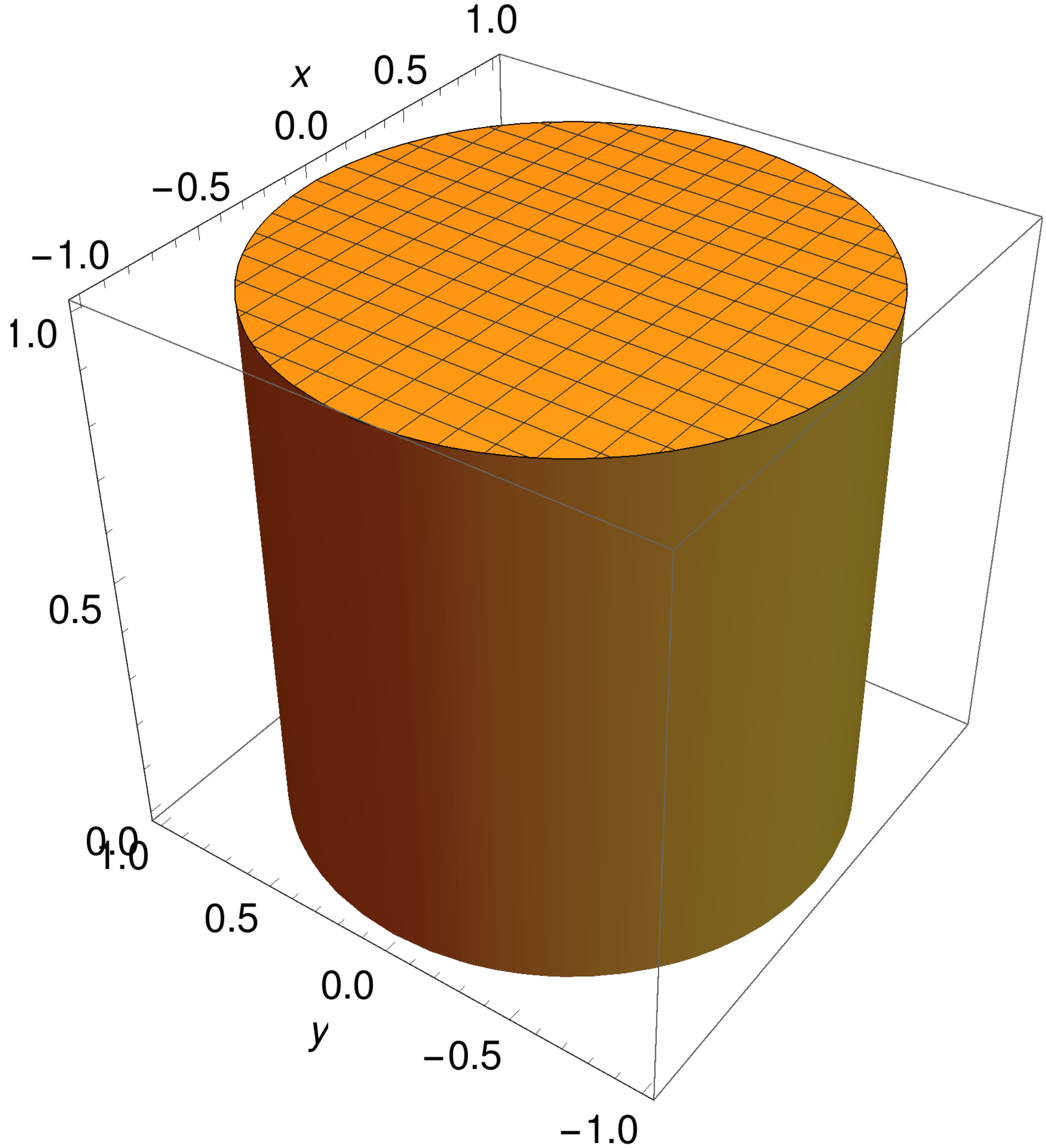
Can you compute this instead?

$$\int_0^1 \int_0^y e^{y^2}\, dx\, dy$$

Double integrals in a circle

Let $D = \{(x, y) : x^2 + y^2 \leq 1\}$. What's the area of

$$\iint_D 1\, dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\, dy\, dx = \int_{-1}^{1} 2\sqrt{1-x^2}\, dx = ??$$

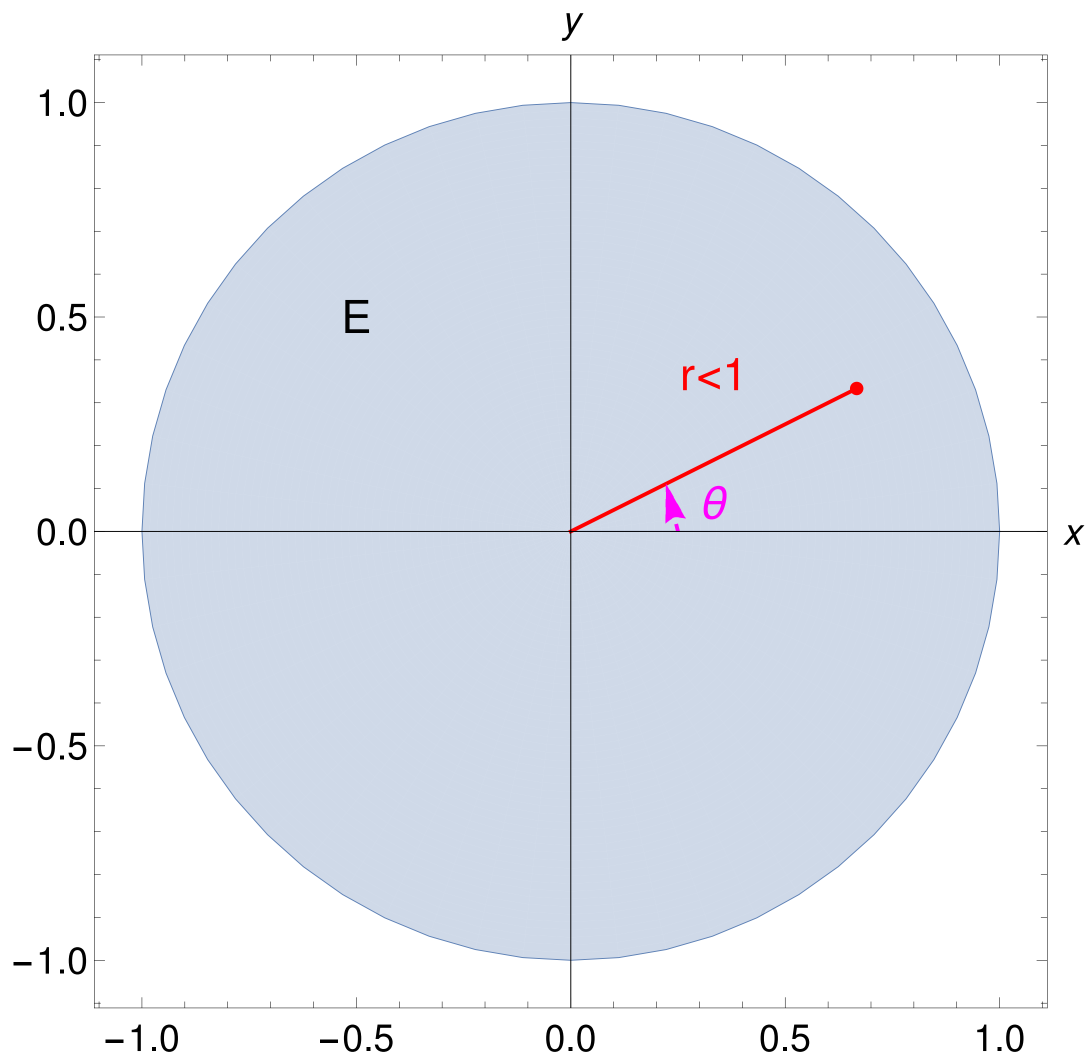


But we know that the answer is π. How can we make the computation easier?

Domains in polar coordinates

The domain $D = \{(x, y) : x^2 + y^2 \leq 1\}$ is much easier to describe in polar coordinates as

$E = \{(r, \theta) : r \leq 1, 0 \leq \theta < 2\pi\}$

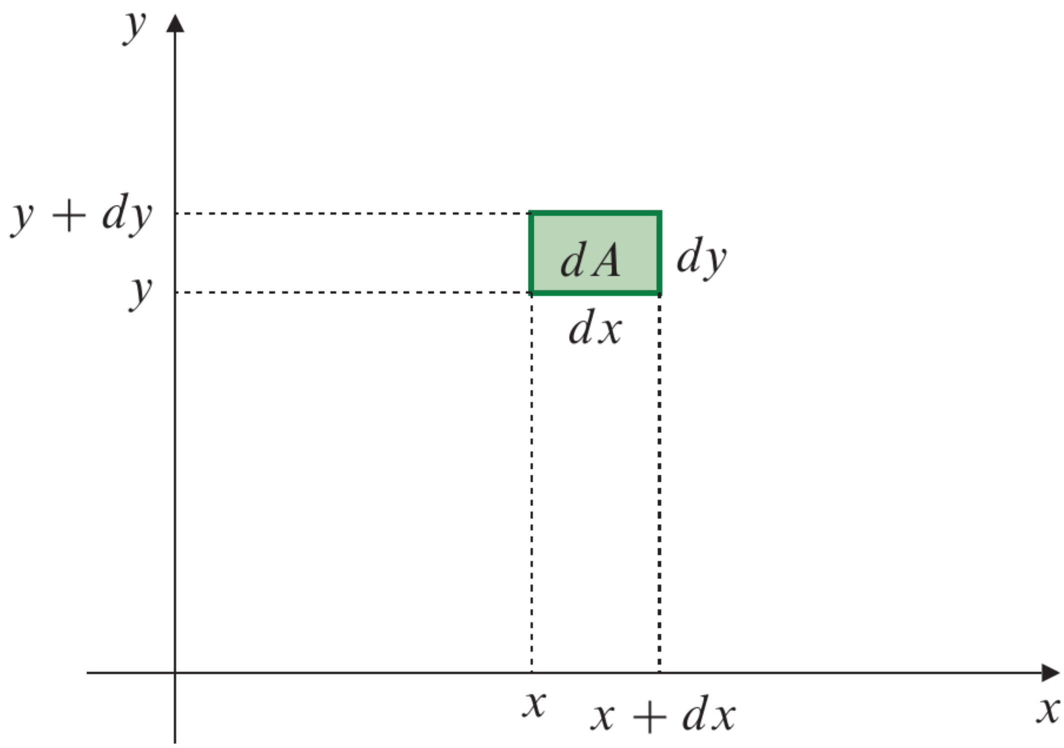


dA in Cartesian coordinates

In Cartesian coordinates, the integral

$$\int\int_D f(x, y)\, dA = \int\int_D f(x, y)\, dx\, dy$$

can be interpreted as "sum" of $f(x, y)$ times dA, an infinitesimal area as shown in the picture.

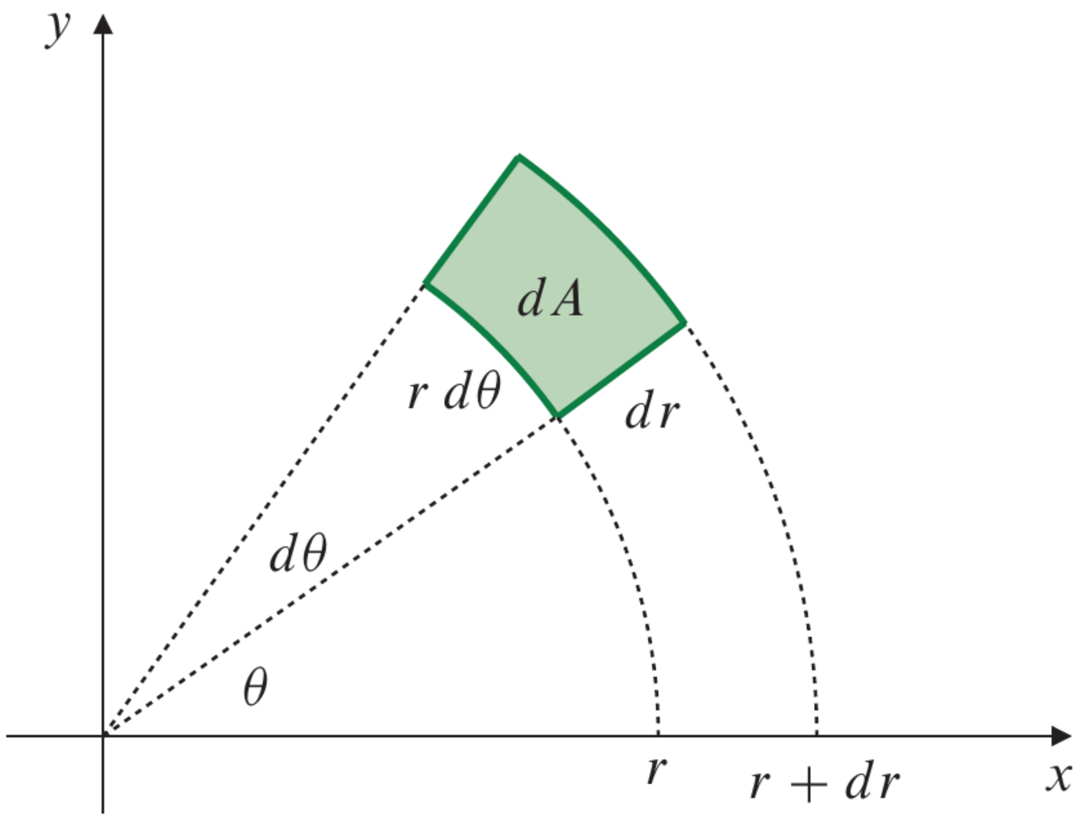


dA in polar coordinates

In polar coordinates, if we want to compute the same “sum”, we need to change dA to

$dA = r\,dr\,d\theta$

as shown in the picture.



Switching to polar coordinates

Therefore, we have

$$\iint_D f(x, y)\, dx\, dy = \iint_E f(r\cos(\theta),\ r\sin(\theta))\, r\, dr\, d\theta$$

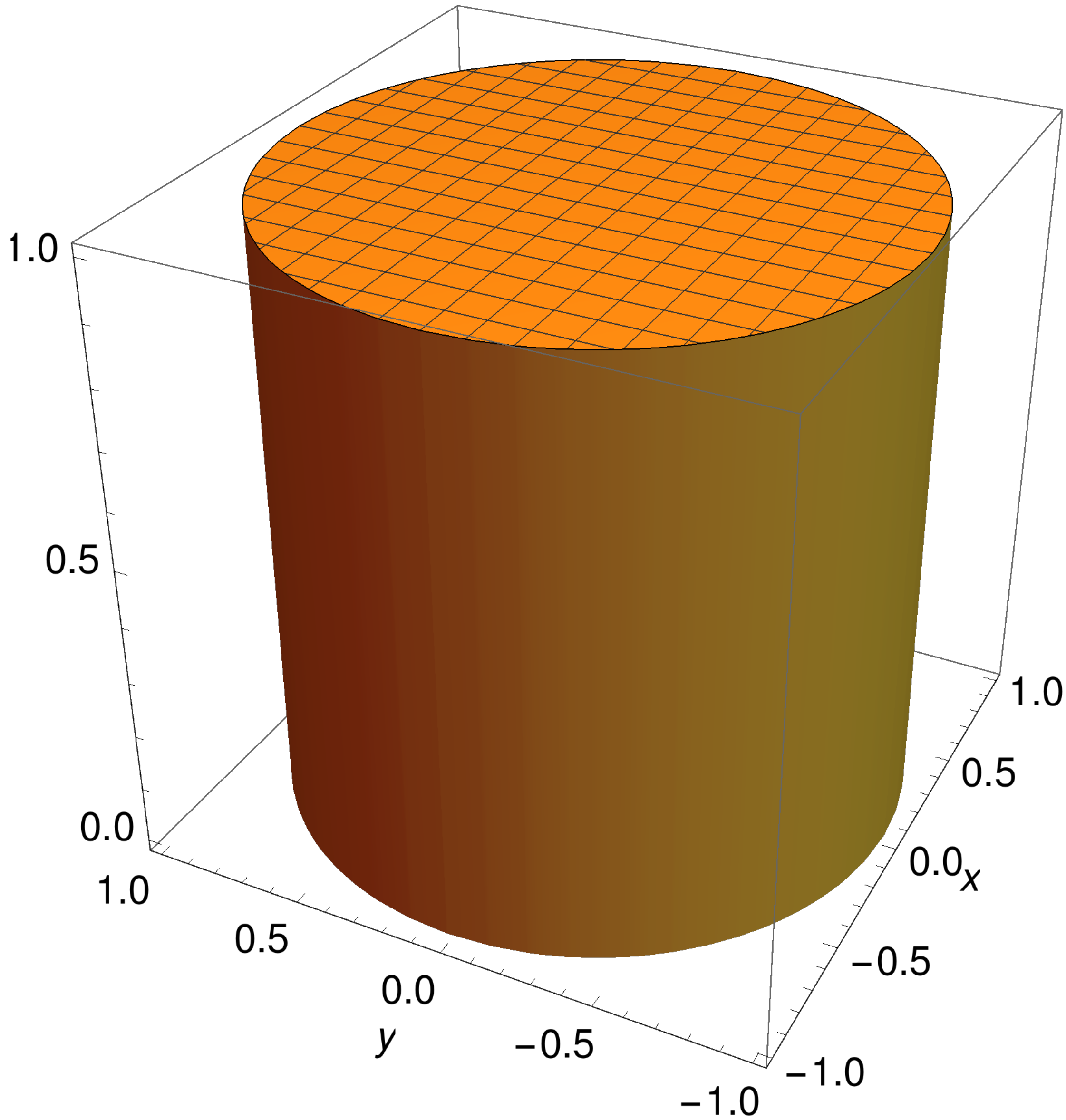
where D, E describe the same region in Cartesian coordinates and polar coordinates respectively.

Example

Let $D = \{(x, y) : x^2 + y^2 \leq 1\}$ and $E = \{(r, \theta) : r \leq 1, 0 \leq \theta < 2\pi\}$. Then

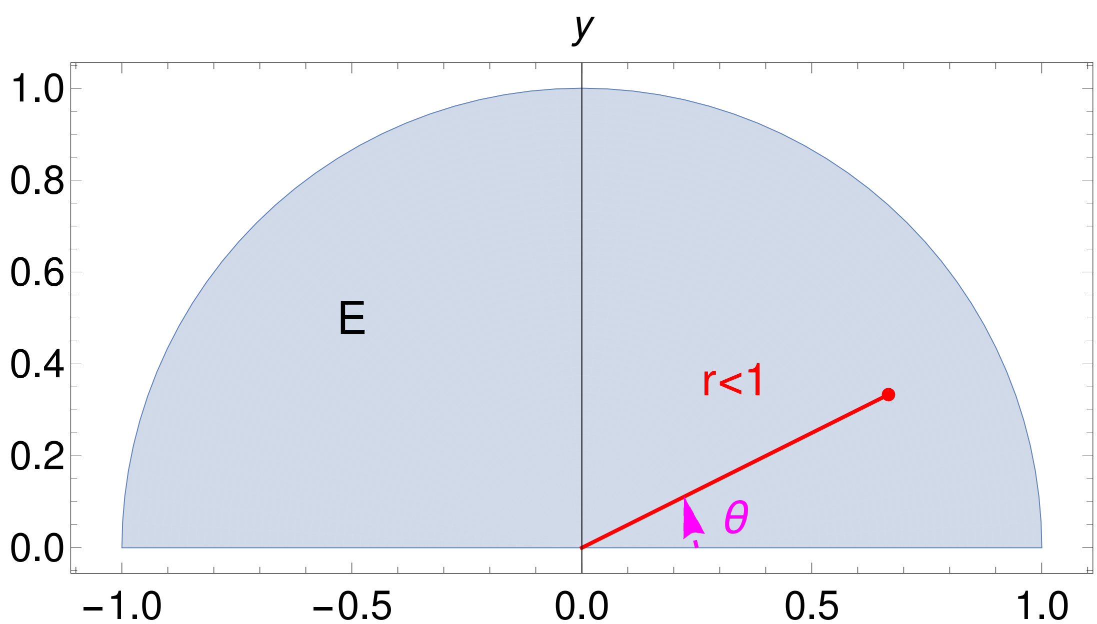
$$\iint_D 1\, dA = \iint_E 1\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 1\, r\, dr\, d\theta = \int_0^{2\pi} \frac{1}{2}\, d\theta = \pi$$


1.0
0.5
0.0
1.0
0.5
0.0
y
−0.5
−1.0
1.0
0.5
0.0
x
−0.5
−1.0

Integral over half of a circle

Let $D = \{(x, y) : x^2 + y^2 \leq 1, y \geq 0\}$ and $E = \{(r, \theta) : r \leq 1, 0 \leq \theta < \pi\}$.



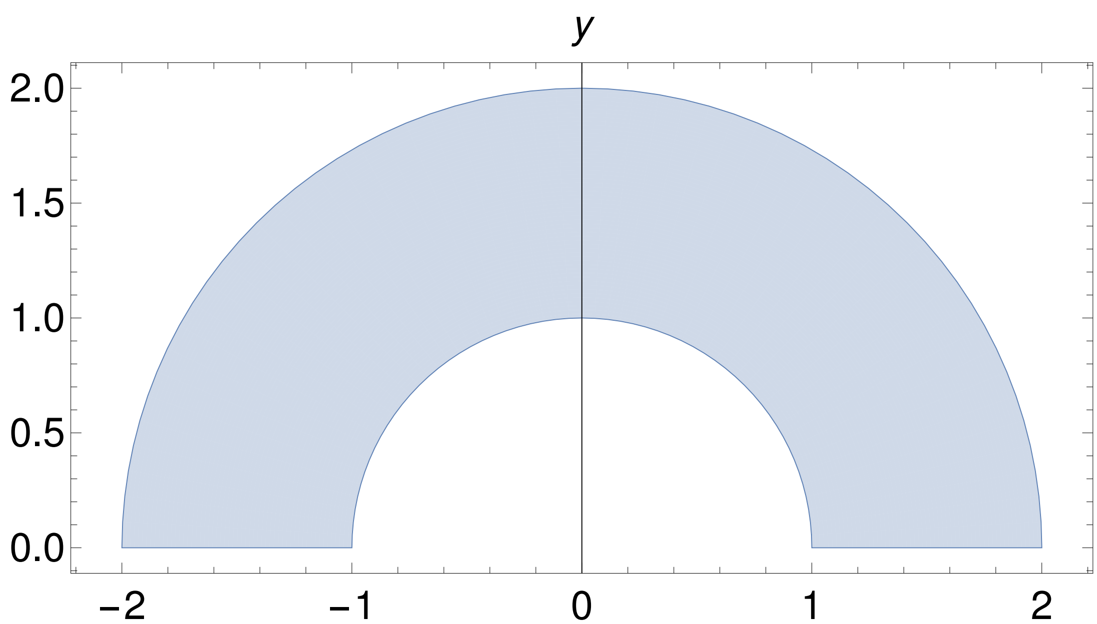
Then

$$\iint_D (1 - x^2 - y^2)\, dA = \iint_E (1 - (r\cos(\theta))^2 - (r\sin(\theta))^2)\, r\, dr\, d\theta$$

$$= \int_0^{\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta = \frac{\pi}{4}$$

Quiz

Let $D = \{(x, y) : 1 \le x^2 + y^2 \le 2, y \ge 0\}$.



Compute

$$\iint_D y \, dA$$

Which one is the correct answer?

$$\frac{14}{9}, \frac{28}{9}, \frac{14}{3}, \frac{56}{9}$$