Lecture 15 —
14.3 Improper Integrals

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Several Variable Calculus, 1MA017, Autumn 2019

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Improper double integral

So far we have only talked about double integral

$$\iint_D f(x, y)\, dA$$

of a bounded function $f(x, y)$ on a bounded domain D.

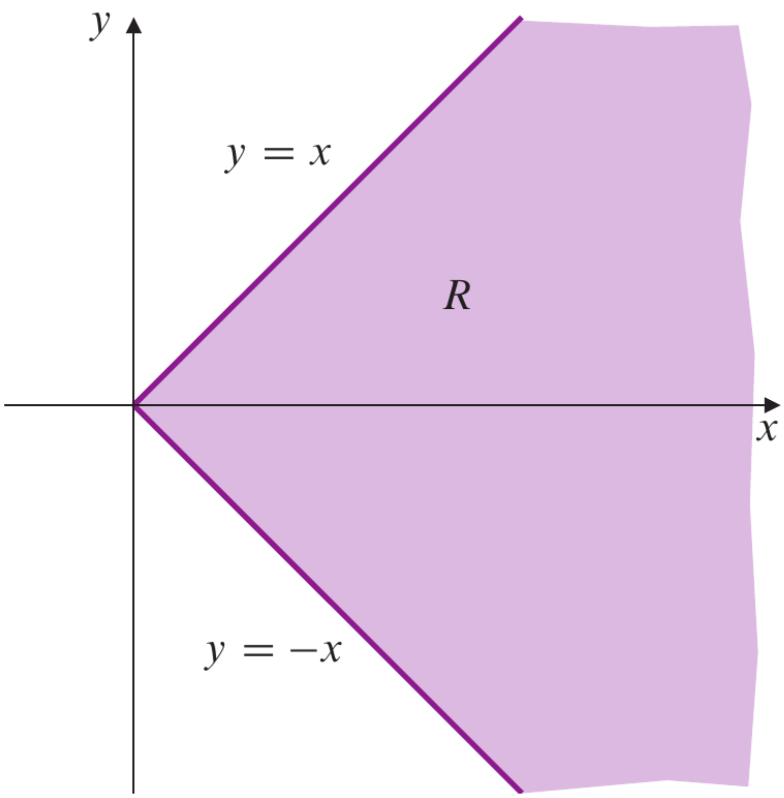
There are two type of integrals we call improper double integrals

- When the function $f(x, y)$ is not bounded
- When the domain D is not bounded.

For $f(x, y) \geq 0$, an improper double integral can either be a finite number (convergence) or infinite (divergence).

Example — unbounded domain

Compute $I = \int\int_D e^{-x^2}\, dA$, where D is the region where $x \geq 0$ and $-x \leq y \leq x$.

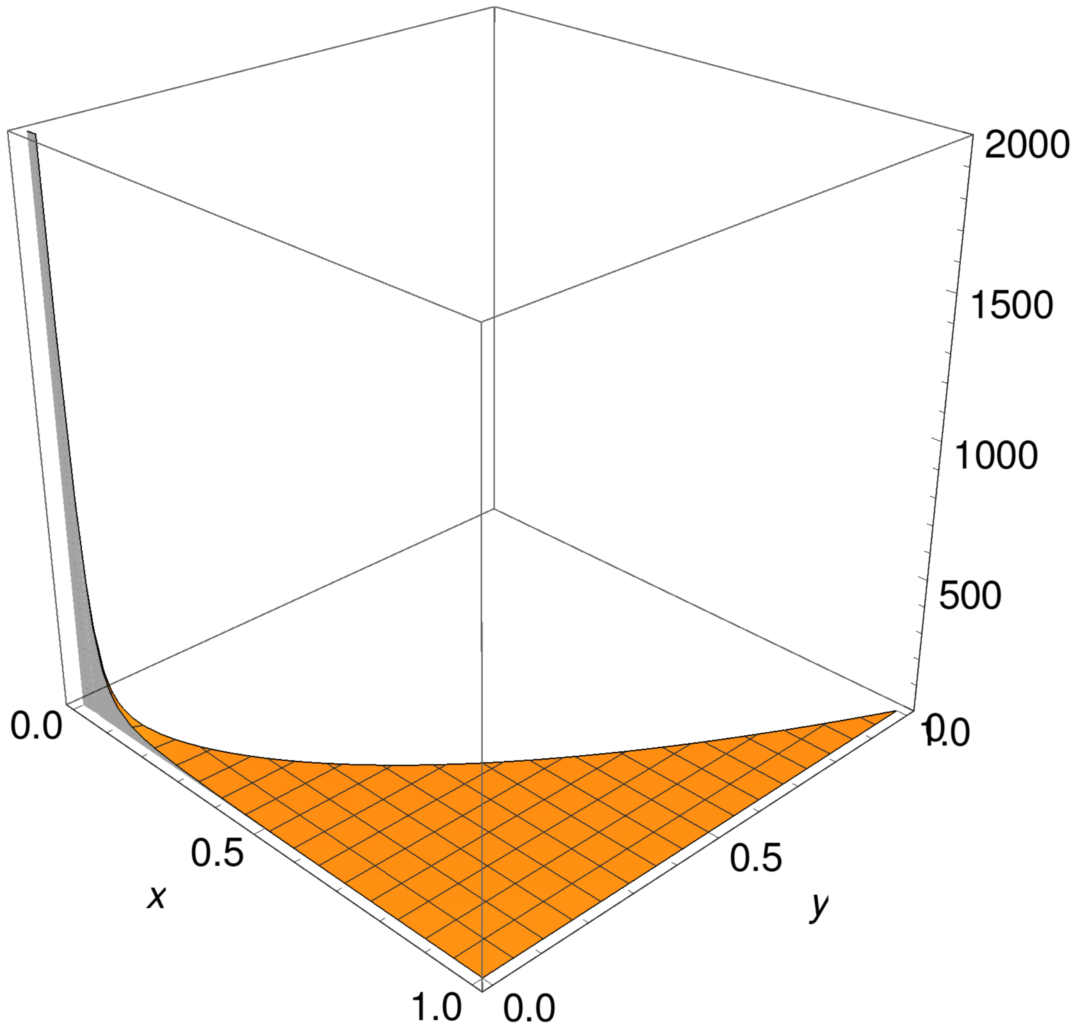


Answer: 1

Example — unbounded function

If D is the region $0 \leq x \leq 1, 0 \leq y \leq x^2$, compute

$$\iint_D \frac{1}{(x+y)^2}\, dA$$



Answer: ln(2).

When iterated integral does not work

If $f(x, y)$ can takes both positive and negative values in the integration area, then iterative integrals may have different value.

Example

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dy \, dx = \frac{\pi}{4}$$

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dx \, dy = -\frac{\pi}{4}$$

Summary — improper double integrals

If the function $f(x, y) \geq 0$ (or $f(x, y) \leq 0$), it is enough to do the iterated integrals to compute the improper integral.

If $f(x, y)$ can takes both positive and negative values in the integration area, then iterated single integration does not tell us anything.

For continuous functions,

$$\int\int_D |f(x, y)| \, dA \text{ is convergent} \Longrightarrow \int\int_D f(x, y) \, dA \text{ is convergent}$$

Quiz

Let D be the region in which $x \geq 0$ and $y \geq 0$. Calculate the improper integral

$$\iint_D e^{-x-y}\, dA$$

The correct answer is

$$\frac{1}{2},\ 1,\ \frac{3}{2},\ 2$$

The gaussian integral

The following is an important integral call the gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

The easiest way to prove this is by computing

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\, dx\, dy$$

Computing this integral in polar coordinates gives the result.

Determine convergence

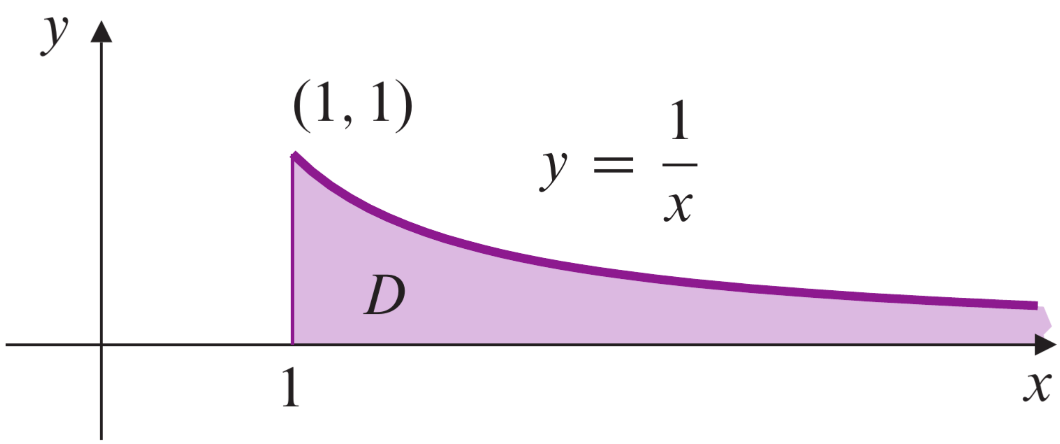
Sometimes there are problems where it's calculate the integral explicitly but can still determine convergence / divergence by comparison with something slightly larger or smaller.

Example

If D is the region as shown in the picture, determine if the double integral

$$\iint_D \frac{1}{x+y}\, dA$$

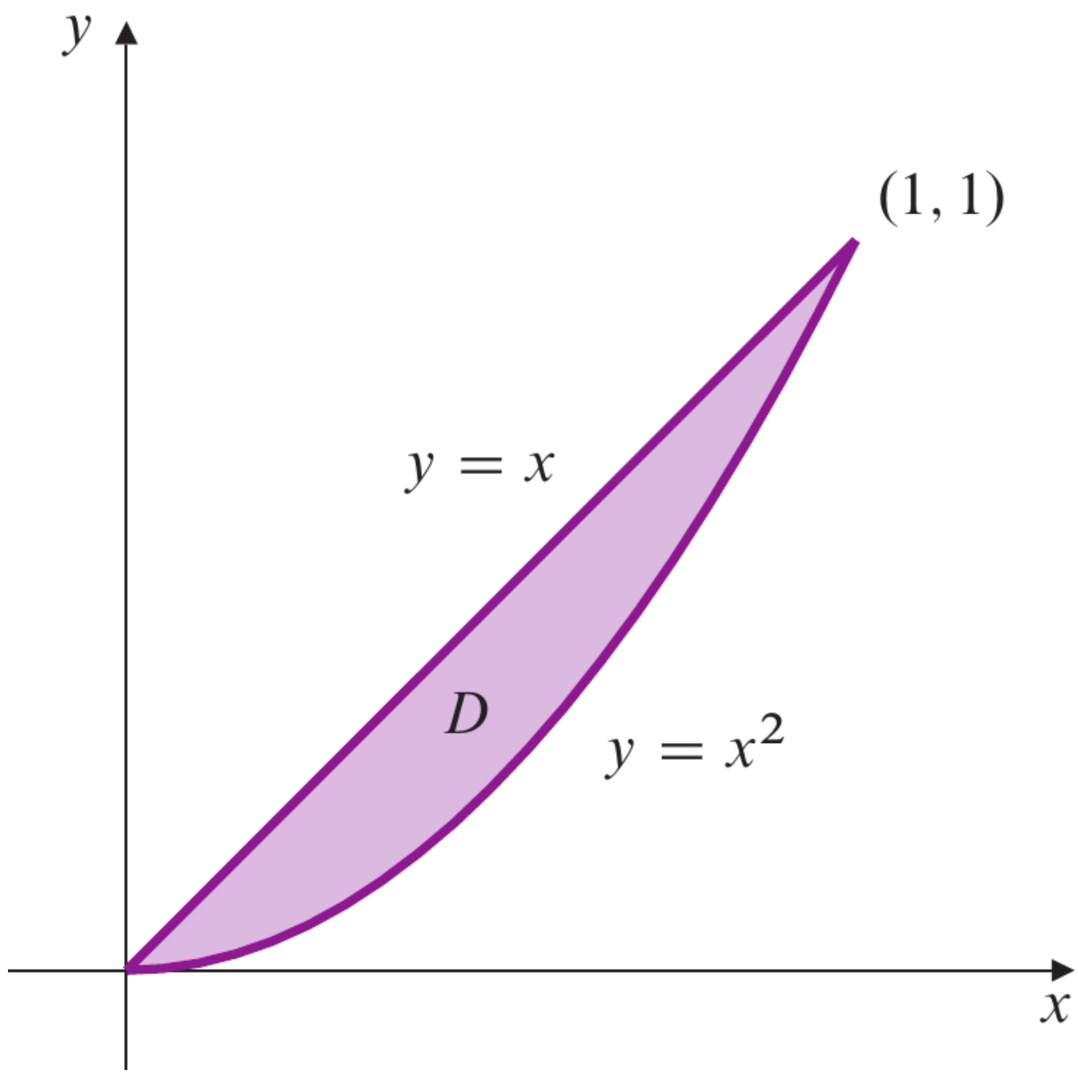
converge.



Example — divergent integral

Let *D* be as in the picture. Show that

$$\int\int_D \frac{1}{x\,y}\,dA = \infty$$



Example — determine convergence

Let $D = \{(x, y) : 0 \le x \le 1, x \le y \le 1\}$. Does the following integral converge?

$$\iint_D \frac{1}{x^3 \sin(x)} \, dA$$

Solution:

Since $\sin(x) \le x$ for $0 \le x \le 1$, we have

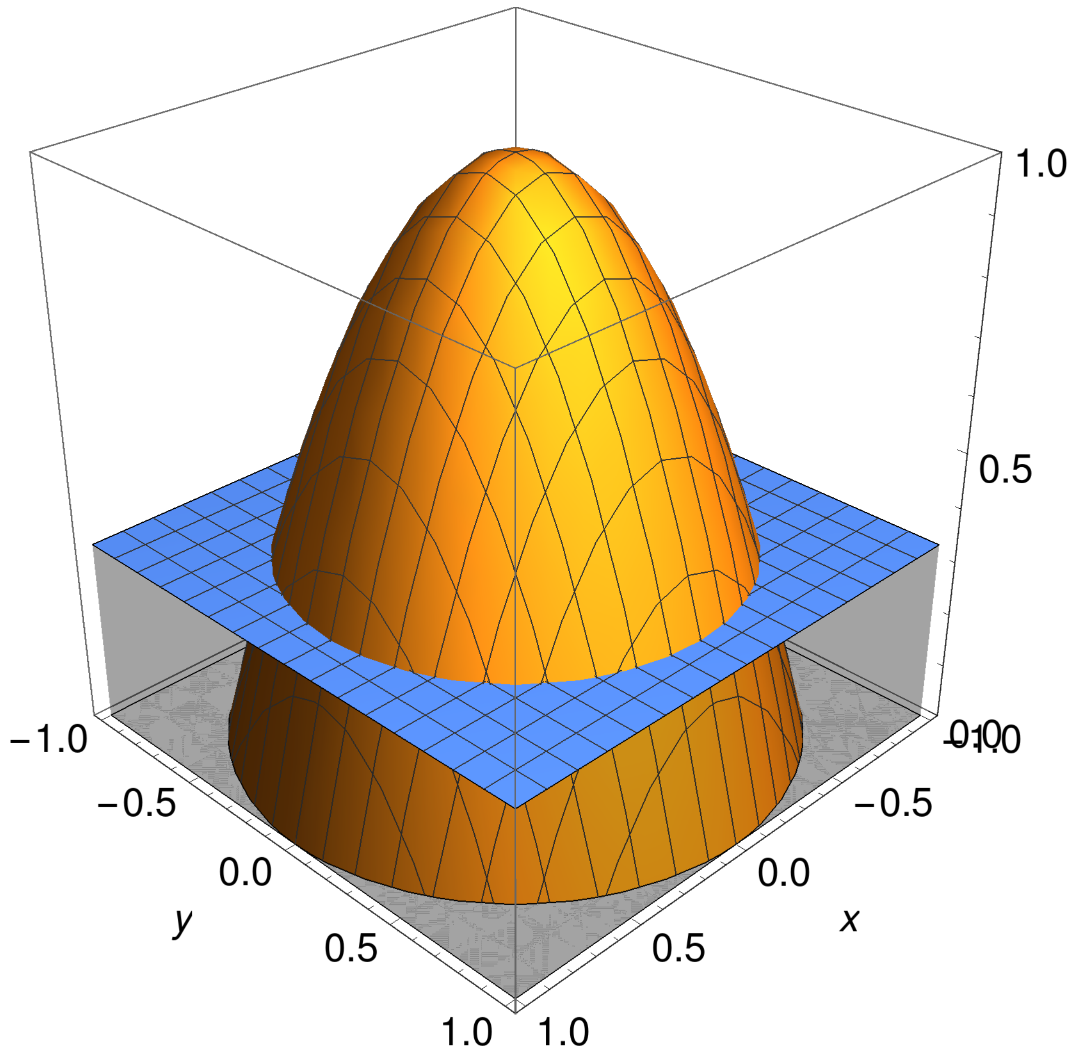
$$\iint_D \frac{1}{x^3 \sin(x)} \, dA \ge \iint_D \frac{1}{x^4} \, dA = \infty$$

So the integral diverges.

A Mean-Value Theorem for double integrals

We define the average of $f(x, y)$ in D by

$$\bar{f} = \frac{1}{\text{area of } D} \times \iint_D f(x, y)\, dA$$



If the function $f(x, y)$ is continuous on a **closed**, **bounded**, **connected** set D in the xy-plane, then there exists a point $(x_0, y_0) \in D$ such that

$$\iint_D f(x, y)\, dA = f(x_0, y_0) \times \text{area of } D$$

Calculate the mass and the average density

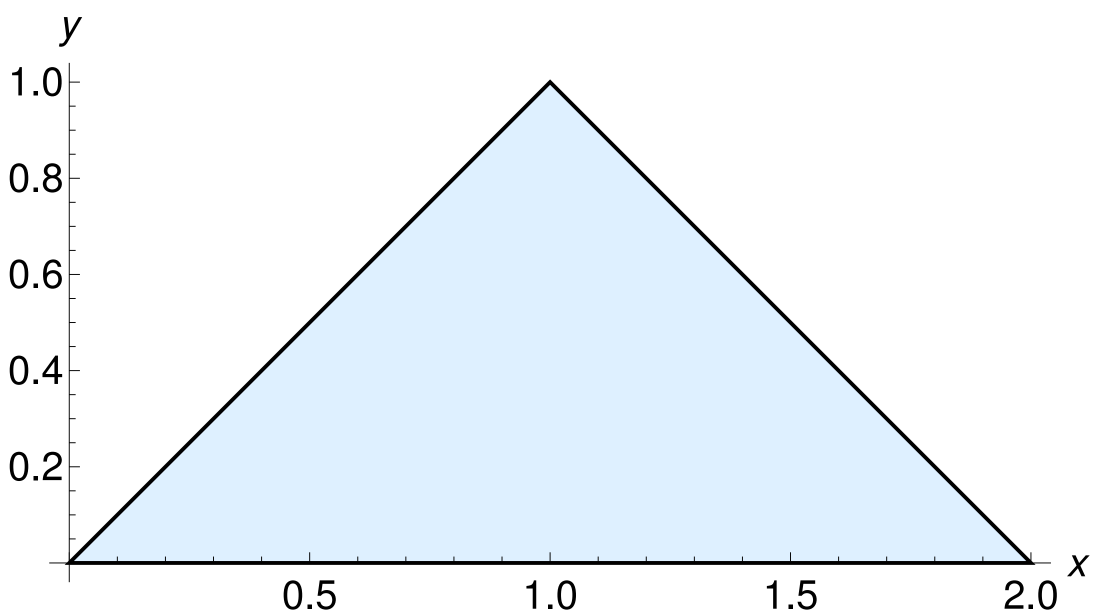
The triangle with vertices in (0, 0), (1, 1) and (2, 0) is covered with a coating whose density at the point (x, y) is given by $f(x, y) = 1 + x$ kg per square meter. Calculate the mass of the coating and the average density of the coating.

Solution

Computing the mass is equivalent to computing the integral

$$\iint_D 1 + x \, dA$$

where D is as shown in the picture.



So the answer is

$$\iint_D 1 + x \, dA = \int_0^1 \int_y^{2-y} (x+1) \, dx \, dy = 2$$

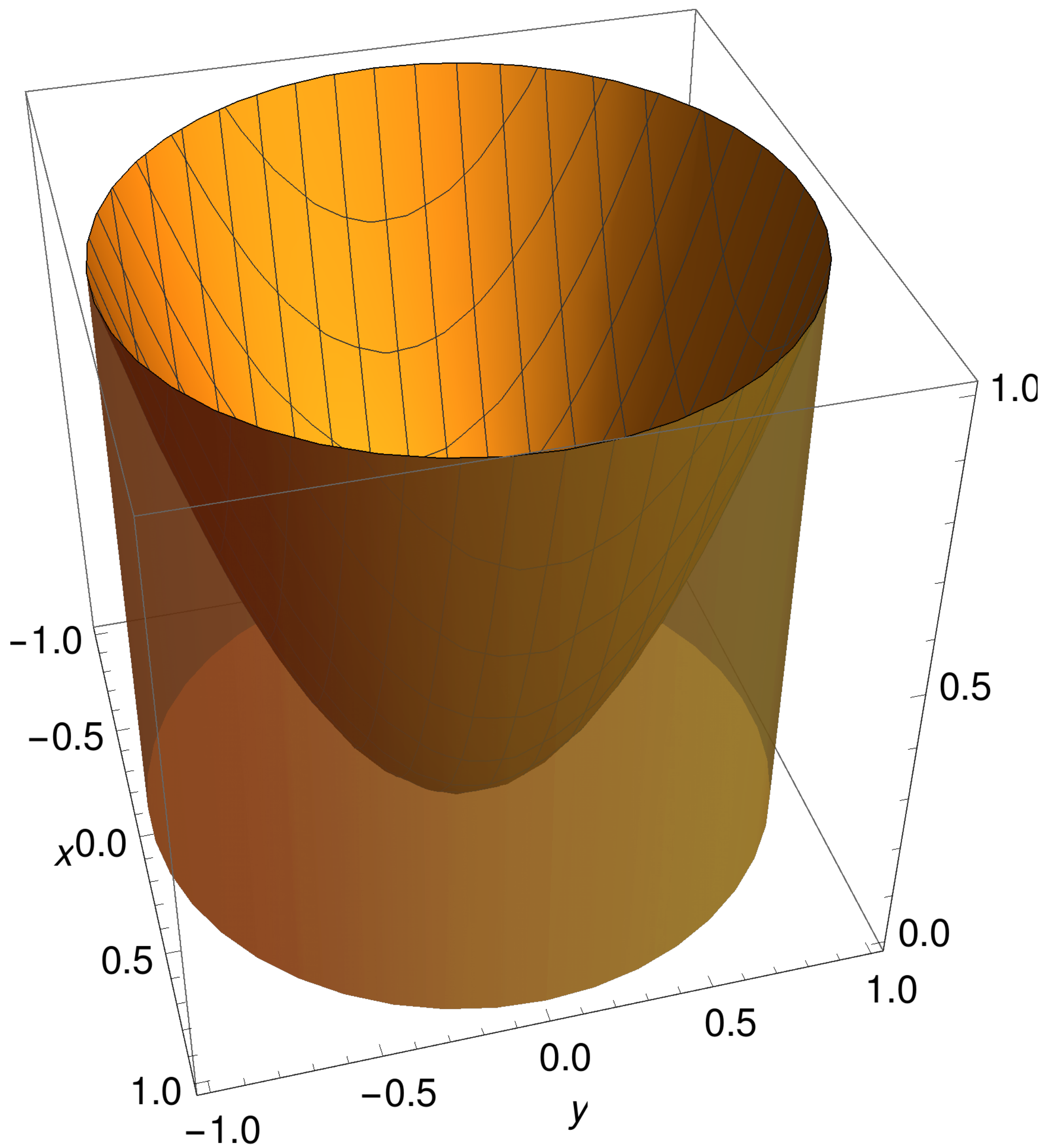
The average density is

$$\frac{1}{\text{area of D}} \times \iint_D 1 + x \, dA = \frac{2}{2} = 1$$

Quiz

Compute the average of $\rho(x, y) = x^2 + y^2$ in the unit circle D.

Which is the correct answer?



Hint Compute the following in polar coordinates.

$$\frac{1}{\pi}\iint_D x^2 + y^2 \, dA = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 r^3 \, dr \, d\theta$$

Which is the correct answer?

$\frac{1}{4}, \frac{1}{2}, 1, 2$