Lecture 16
14.5—14.7 Triple Integrals, Applications

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The calories in a cake

A cake occupies the domain K in 3-dimensional space. The density of calorie in the cake is given by the function $\rho(x, y, z)$. How can we compute the total calories contained in the cake?

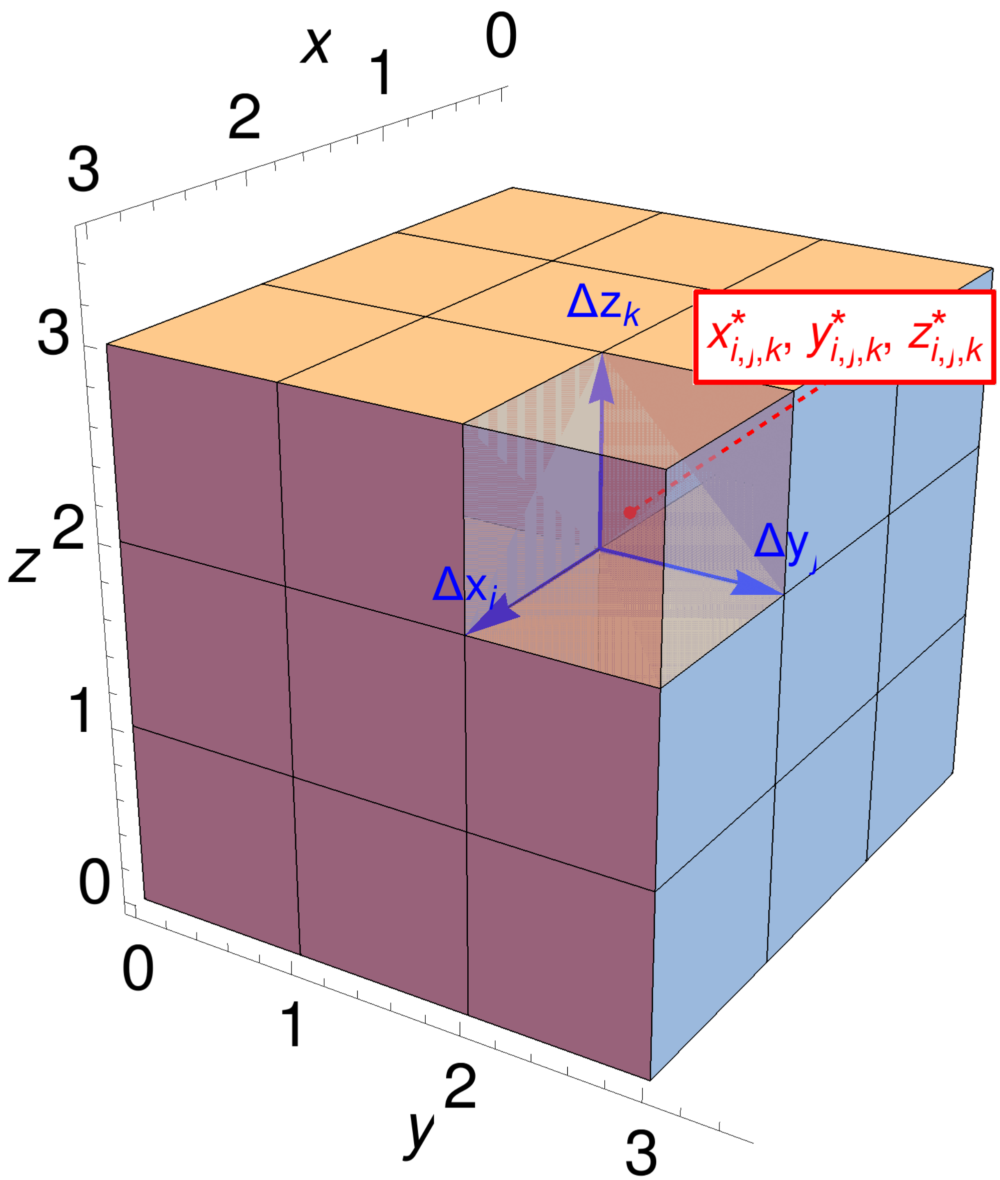
What is a triple integral

We define triple integrals with a Riemann sums in a way similar to double integrals

$$\iiint_K f(x, y, z)\, dV = \iiint_K f(x, y, z)\, dx\, dy\, dz$$

$$= \lim \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} f(x^*_{i,j,k}, y^*_{i,j,k}, z^*_{i,j,k})\, \Delta x_i\, \Delta y_j\, \Delta z_k$$

where the limit is taken when $\max(\Delta x_i, \Delta y_i, \Delta z_k) \to 0$.



Interpretations of triple integral

Integrating 1 over K gives

$$\text{Volume of } K = \iiint_K 1\, dV$$

Integrating the density $\rho(x, y, z)$ gives

$$\text{mass of } K = \iiint_K \rho(x, y, z)\, dV$$

The centre of mass of K is given by

$$(\overline{x}, \overline{y}, \overline{z}) = \frac{1}{\text{mass of } K}\left(\iiint_B x\, \rho\, dV,\ \iiint_B y\, \rho\, dV,\ \iiint_B z\, \rho\, dV\right)$$

Compute triple integral

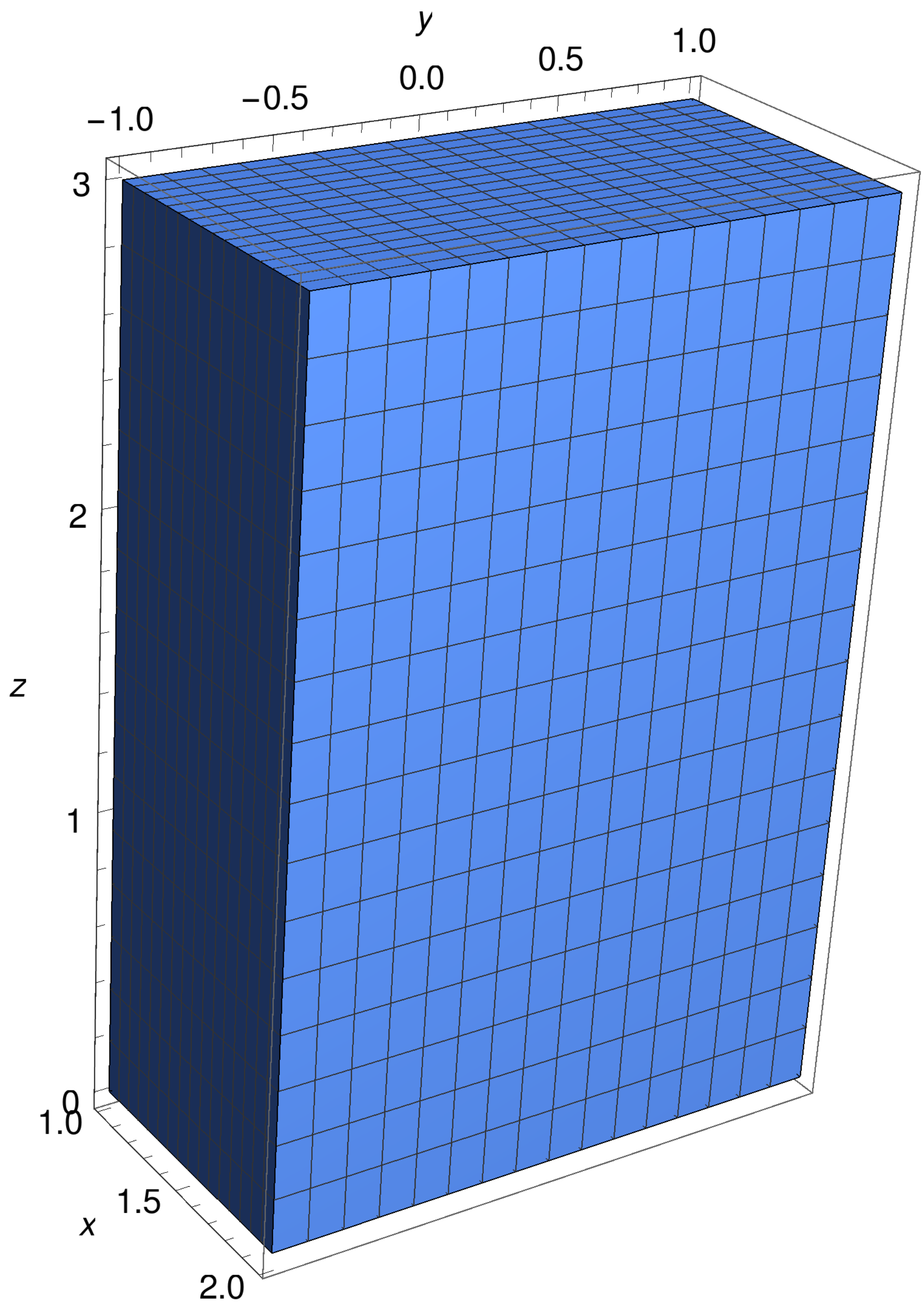
Calculation of triple integrals can be done as iterated simple integrals.

Example Compute the triple integral over K in which $1 \leq x \leq 2$, $-1 \leq y \leq 1$, and $0 \leq z \leq 3$ can be computed as

$$\iiint_K f(x, y, z)\, dV = \int_0^3 \int_{-1}^1 \int_1^2 f(x, y, z)\, dx\, dy\, dz$$

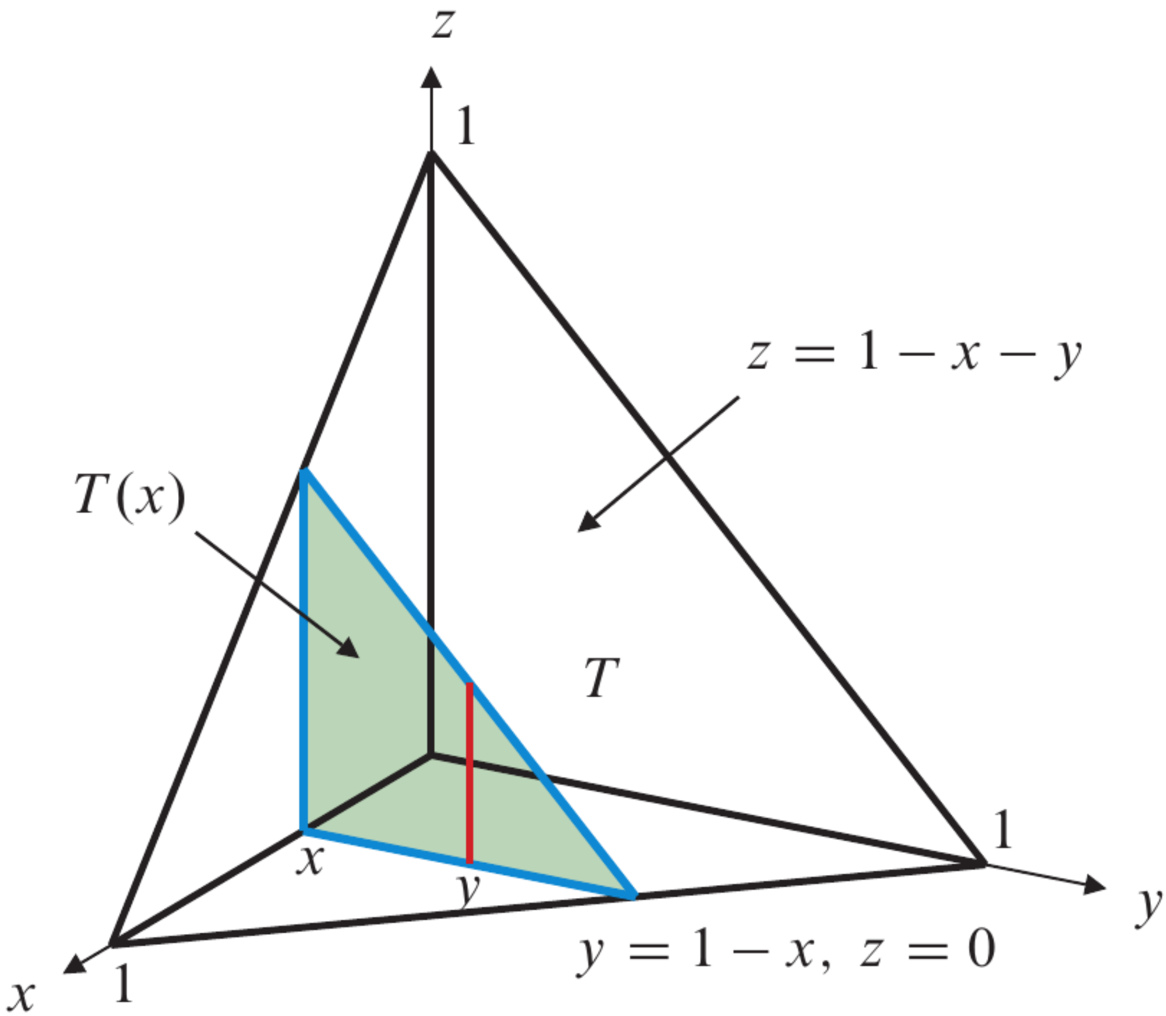
Thus the volume of K is

$$\int_0^3 \int_{-1}^1 \int_1^2 1\, dx\, dy\, dz = 6$$


y
−1.0
−0.5
0.0
0.5
1.0
z
3
2
1
0
x
1.0
1.5
2.0

When integration domain is not a box

Let T be the tetrahedron shown in the picture. Compute $I = \int\int\int_T f(x, y, z)\, dV$



Answer:

$$\iiint_T f(x, y, z)\, dV = \int_0^1 \left(\iint_{T(x)} f\, dz\, dy \right) dx = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} f\, dz\, dy\, dx$$

Quiz

We want to compute the following triple integral over $K = \{0 \leq x \leq 1, 0 \leq y \leq 2, 0 \leq z \leq 3\}$

$$\iiint_K x\, y\, dV$$

Hint Compute the following iterated integral

$$\int_0^1 \int_0^2 \int_0^3 x\, y\, dz\, dy\, dx$$

Review: Cylindrical coordinates

To go from cylindrical coordinates to Cartesian coordinates, we use the equations

$x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$.

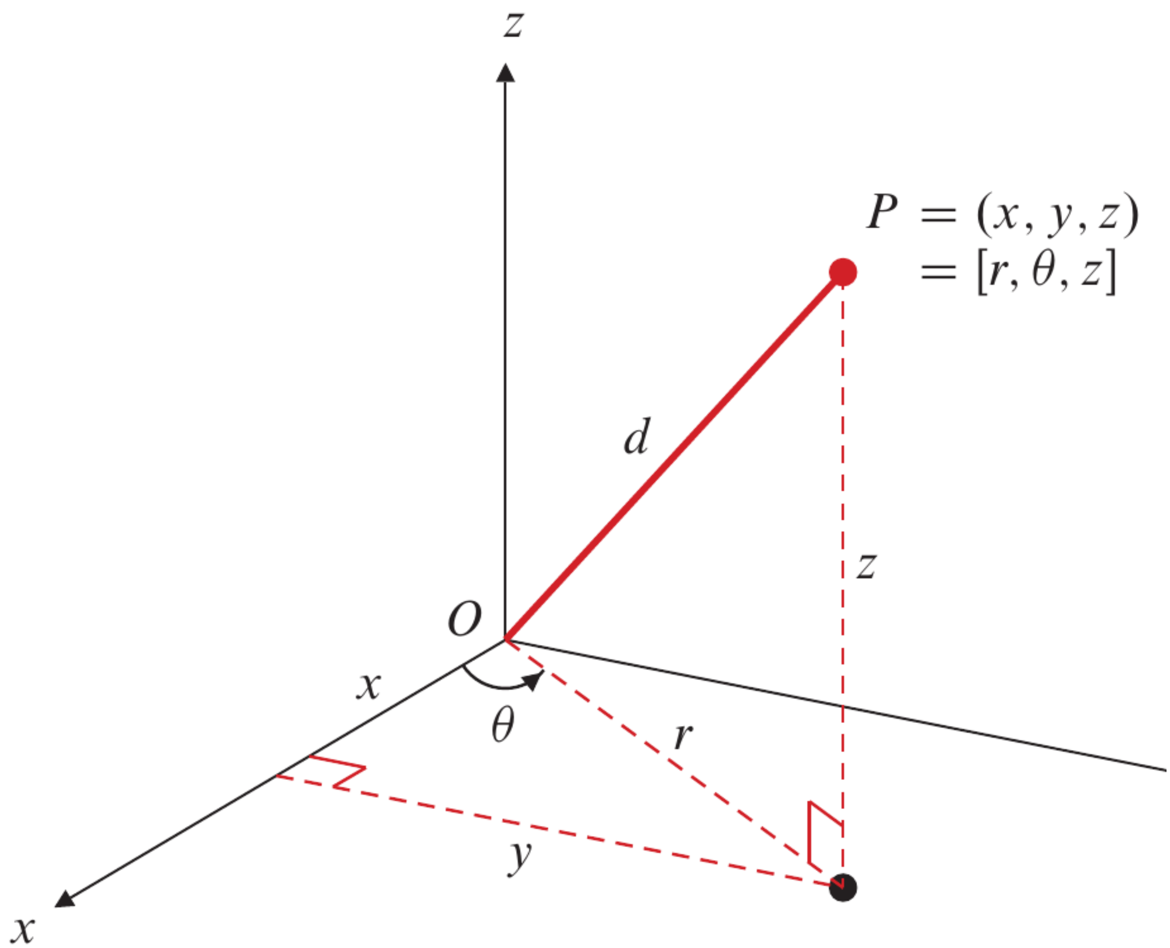
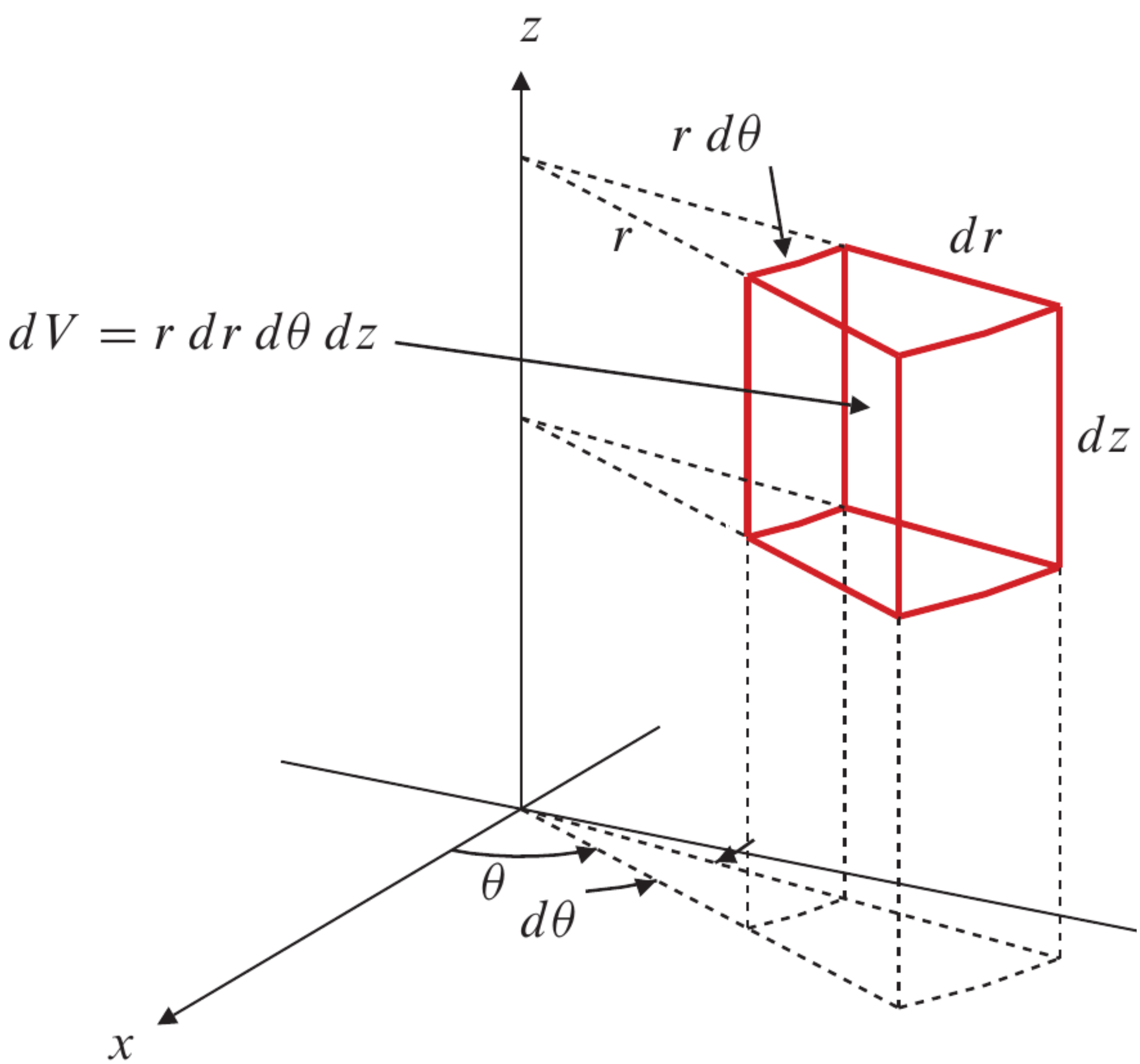


Figure 14.45 The cylindrical coordinates of a point

Integrate in cylindric coordinates

As shown in the picture, when we integrate in cylindric coordinates, we should replace $dx\,dy\,dz$ by $r\,dr\,d\theta\,dz$. Thus

$$\iiint_K f(x, y, z)\,dx\,dy\,dz = \iiint_F f(r\cos(\theta), r\sin(\theta), z)\,r\,dr\,d\theta\,dz$$

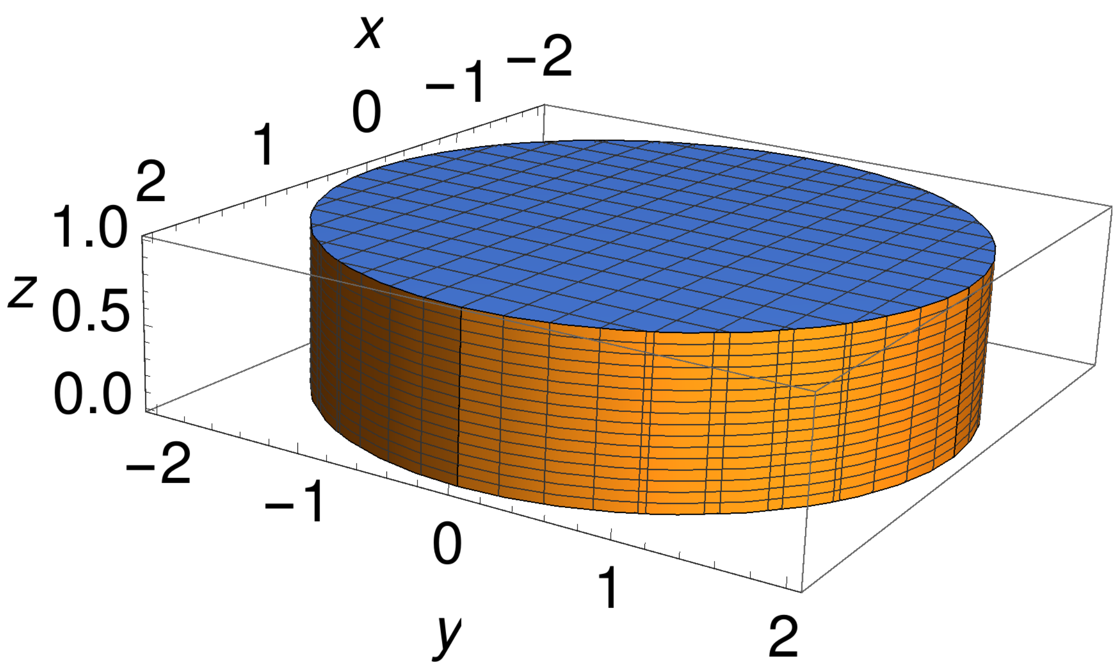


Example

Compute

$$\iiint_K x^2 + y^2 + z^2 \, dx \, dy \, dz$$

over K in which $x^2 + y^2 \leq 4, 0 \leq z \leq 1$



Answer

$$\frac{28\,\pi}{3}$$

Review: Spherical coordinates

To go from spherical coordinates to Cartesian coordinates, we use

$x = R \sin(\phi) \cos(\theta)$, $y = R \sin(\phi) \sin(\theta)$, $z = R \cos(\phi)$.

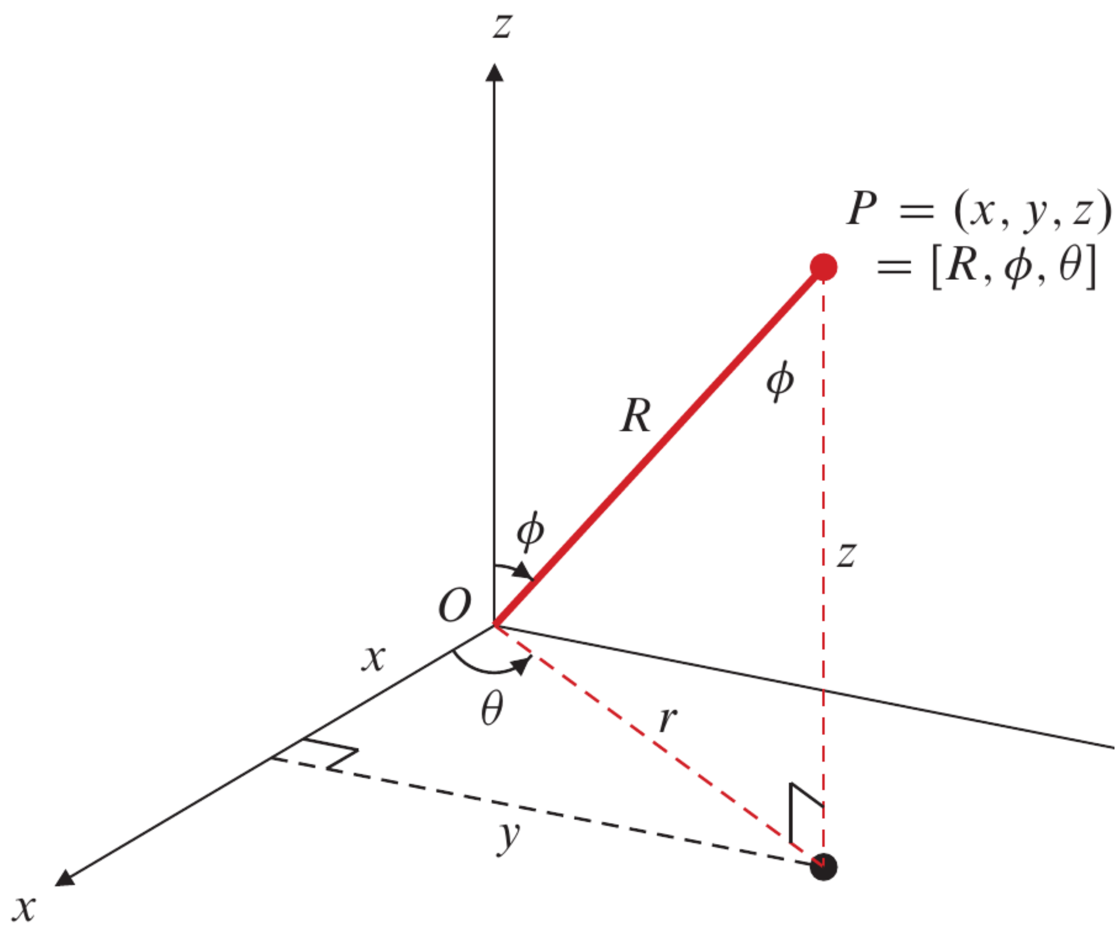
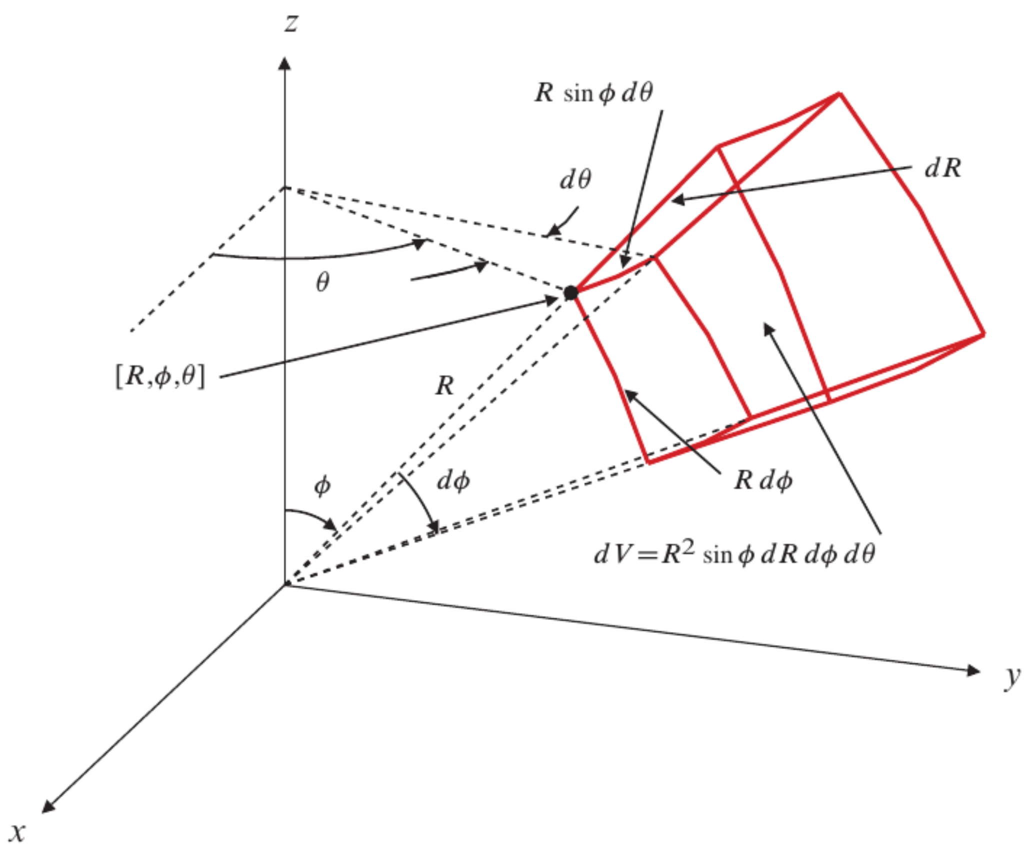


Figure 14.50 The spherical coordinates of a point

Integrate in spherical coordinates

As shown in the picture, when we integrate in cylindric coordinates, we should replace $dx\,dy\,dz$ by $R^2 \sin(\phi)\,dR\,d\theta\,d\phi$. Thus

$$\iiint_K f(x, y, z)\,dx\,dy\,dz = \iiint_F f(R\sin(\phi)\cos(\theta), R\sin(\phi)\sin(\theta), R\cos(\phi))\,R^2\sin(\phi)\,dR\,d\theta\,d\phi$$

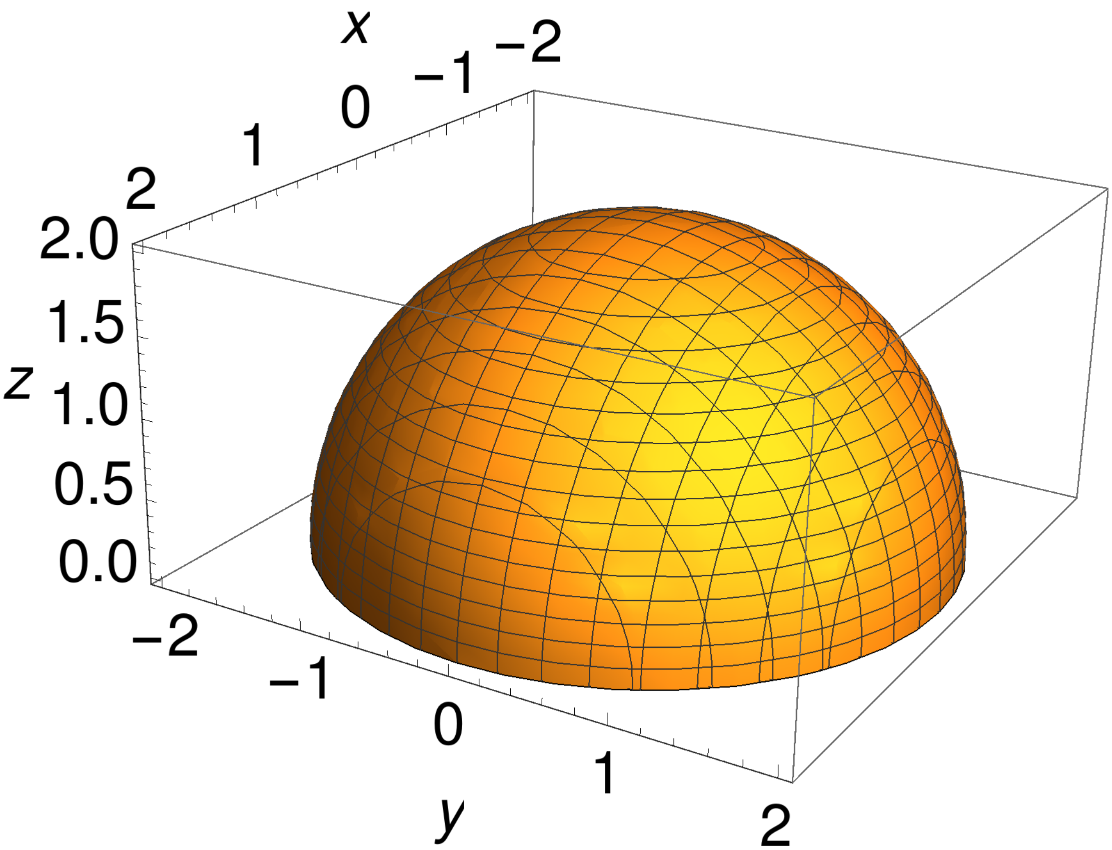


Example

Compute

$$\iiint_K z \, dx \, dy \, dz$$

over K in which $x^2 + y^2 + z^2 \leq 4, 0 \leq z$

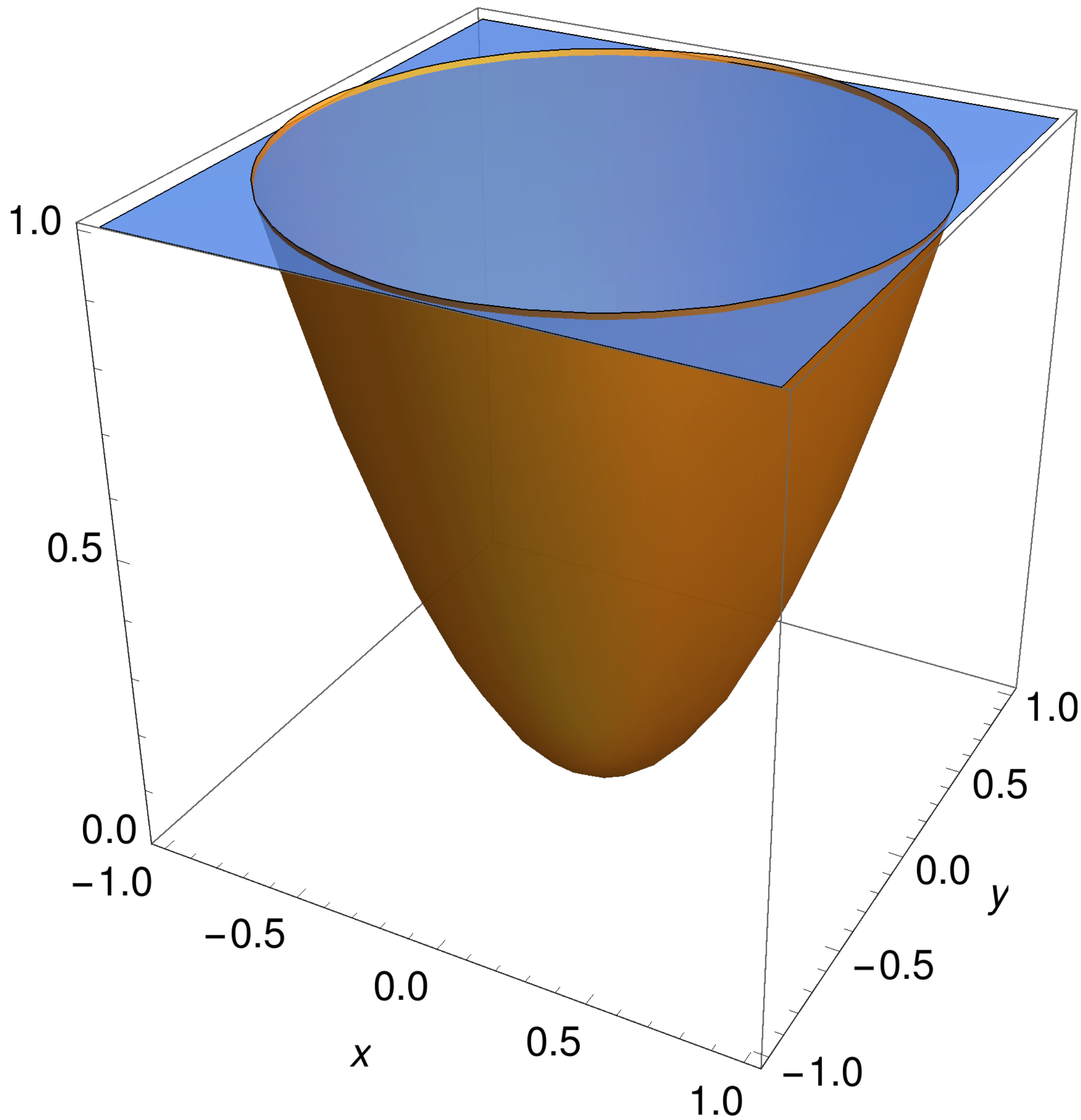


Answer

$$2\pi$$

Quiz

Consider K enclosed between $z = x^2 + y^2$ and $z = 1$.



Compute the integral

$$\iiint_K z \, dx \, dy \, dz$$

Hint: In cylindrical coordinates, the integration domain is

$$F = \left\{(r, \theta, z) : 0 \le z \le 1,\ 0 < \theta \le \frac{\pi}{2},\ 0 \le r \le \sqrt{z}\right\}$$

So we only need to compute the following iterative integral

$$\int_0^1 \int_0^{2\pi} \int_0^{\sqrt{z}} z\, r \, dr \, d\theta \, dz$$