Lecture 17 – Vector Fields

Several Variable Calculus, 1MA017

Xing Shi Cai

Autumn 2019

Department of Mathematics, Uppsala University, Sweden

Summary

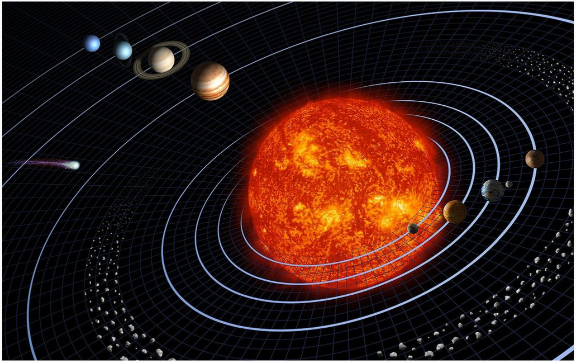
Please watch this video before the lecture: 18

Today we will talk about

- 15.1 Vector and Scalar Fields
- 15.2 Conservative Fields

Vector fields

Gravity holds the solar system together



Example of vector field – Gravitational field

Every position around a point mass P_0 gets a gravitational force, which has a direction and magnitude. We call these forces a gravitational field.

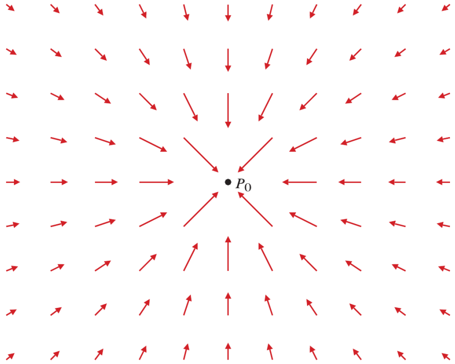
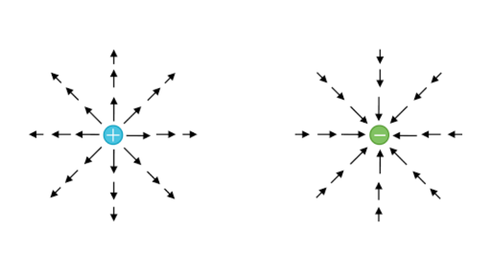


Figure 15.1 The gravitational field of a point mass located at P_0

Example of vector field – Electric field

An electric field $\mathbf{E}(x, y, z)$ surrounds an electric charge, and exerts force on other charges at (x, y, z), attracting or repelling them.



Electric field

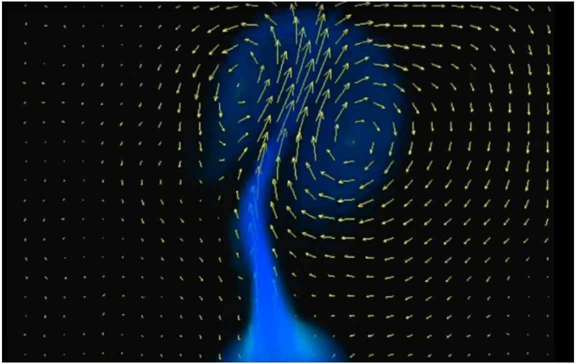
Example of vector field – Electric field

An electric field $\mathbf{E}(x, y, z)$ surrounds an electric charge, and exerts force on other charges at (x, y, z), attracting or repelling them.

Electric field around a cat

Example of vector field – Velocity field

A velocity field $\mathbf{v}(x, y, z)$ of a moving fluid is the velocity of the partial at (x, y, z).



Velocity field of fluid

What is a vector field

A vector field is simply a function $\mathbf{F} : \mathbb{R}^3 \mapsto \mathbb{R}^3$. We write such a function as

$$\mathbf{F}(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z)).$$

What is a vector field

A vector field is simply a function $\mathbf{F} : \mathbb{R}^3 \mapsto \mathbb{R}^3$. We write such a function as

$$\mathbf{F}(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z)).$$

This can be seen as if at each point (x, y, z) there is a vector given by $\mathbf{F}(x, y, z)$.

What is a vector field

A vector field is simply a function $\mathbf{F} : \mathbb{R}^3 \mapsto \mathbb{R}^3$. We write such a function as

$$\mathbf{F}(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z)).$$

This can be seen as if at each point (x, y, z) there is a vector given by $\mathbf{F}(x, y, z)$.

We will only talk about smooth vector fields, i.e., the case P, Q, R have partial derivatives of all orders.

What is a vector field

A vector field is simply a function $\mathbf{F} : \mathbb{R}^3 \mapsto \mathbb{R}^3$. We write such a function as

$$\mathbf{F}(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z)).$$

This can be seen as if at each point (x, y, z) there is a vector given by $\mathbf{F}(x, y, z)$.

We will only talk about smooth vector fields, i.e., the case P, Q, R have partial derivatives of all orders.

A planar vector field is simply a function $\mathbf{F} : \mathbb{R}^2 \mapsto \mathbb{R}^2$. We write such a function as

$$\mathbf{F}(x, y) = (P(x, y), Q(x, y))$$

Gravitational field

The gravitational force field due to a point mass m located at position $(0,0,0)$ is

$$\mathbf{F}(x,y,z) = -\frac{km}{|(x,y,z)|^3}(x,y,z)$$

where $k \approx 6.674$ is the gravitational constant.

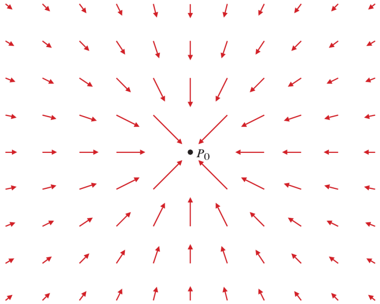


Figure 15.1 The gravitational field of a point mass located at P_0

Example of vector field – Rotating solid body

The velocity field of a solid disc rotating around its center with angular velocity Ω is

$$\mathbf{v}(x, y) = (-\Omega y, \Omega x).$$

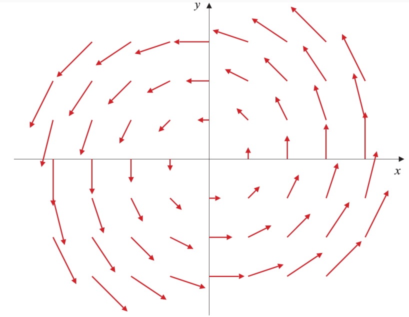


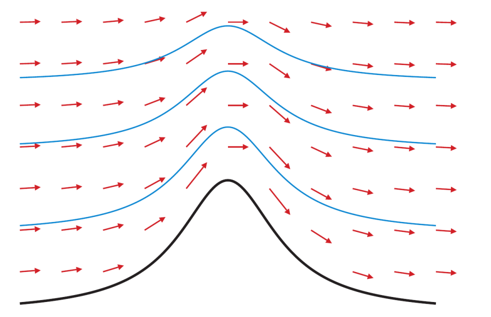
Figure 15.2 The velocity field of a rigid body rotating about the z-axis

Field lines

Field lines

A curve which is tangent to a vector field everywhere is call a field line.

In a velocity field, field lines are trajectories of moving particles.



Finding field lines

If a field line can be parametrized as $\mathbf{r}(t)$, then

$$\mathbf{r}'(t) = \lambda(t)\mathbf{F}(\mathbf{r}(t)),$$

for some function $\lambda : \mathbb{R} \mapsto \mathbb{R}$.

Finding field lines

If a field line can be parametrized as $\mathbf{r}(t)$, then

$$\mathbf{r}'(t) = \lambda(t)\mathbf{F}(\mathbf{r}(t)),$$

for some function $\lambda : \mathbb{R} \mapsto \mathbb{R}$.

For a vector field $\mathbf{F}(x, y, z) = (P, Q, R)$, the above implies that

$$\frac{\mathrm{d}x}{P(x,y,z)} = \frac{\mathrm{d}y}{Q(x,y,z)} = \frac{\mathrm{d}z}{R(x,y,z)}$$

Finding field lines

If a field line can be parametrized as $\mathbf{r}(t)$, then

$$\mathbf{r}'(t) = \lambda(t)\mathbf{F}(\mathbf{r}(t)),$$

for some function $\lambda : \mathbb{R} \mapsto \mathbb{R}$.

For a vector field $\mathbf{F}(x, y, z) = (P, Q, R)$, the above implies that

$$\frac{\mathrm{d}x}{P(x, y, z)} = \frac{\mathrm{d}y}{Q(x, y, z)} = \frac{\mathrm{d}z}{R(x, y, z)}$$

and for a plane vector field $\mathbf{F}(x, y) = (P, Q)$,

$$\frac{\mathrm{d}x}{P(x, y)} = \frac{\mathrm{d}y}{Q(x, y)}$$

Example

Find the field lines of the gravitational field

$$\mathbf{F}(x, y, z) = -\frac{km}{|(x, y, z)|^3}(x, y, z)$$

Answer: Any lines that goes through $(0, 0, 0)$.

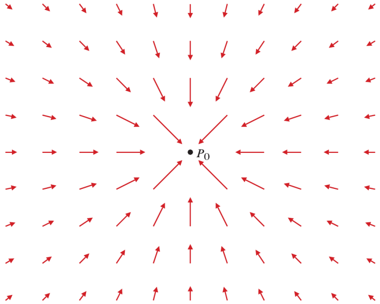


Figure 15.1 The gravitational field of a point mass located at P_0

Example

Find the field lines of

$$\mathbf{F}(x, y, z) = (xz, 2x^2z, x^2)$$

Answer: $y = x^2 + C_1$ and $y = z^2 + C_2$.

Quiz – rotating solid body

Find the field lines of $\mathbf{H}(x, y) = (-\Omega y, \Omega x)$, where Ω is a constant.

Hint: Solve the equation that

$$\frac{\mathrm{d}x}{-\Omega y} = \frac{\mathrm{d}y}{\Omega x}$$

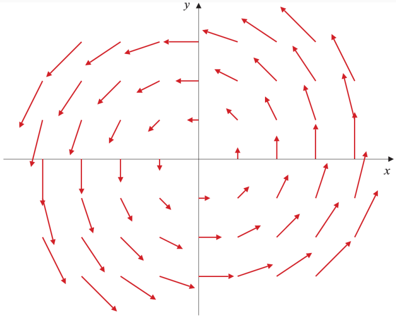


Figure 15.2 The velocity field of a rigid body rotating about the z-axis

Conservative vector fields

Conservative vector field

If there is a function $\varphi : \mathbb{R}^3 \mapsto \mathbb{R}$ such that $\nabla\varphi = \mathbf{F}$, then $\mathbf{F}$ is a conservative vector field. φ is a potential for $\mathbf{F}$.

Conservative vector field

If there is a function $\varphi : \mathbb{R}^3 \mapsto \mathbb{R}$ such that $\nabla\varphi = \mathbf{F}$, then $\mathbf{F}$ is a conservative vector field. φ is a potential for $\mathbf{F}$.

Example

Let $\varphi(x, y) = \arctan(xy)$. Let

$$\mathbf{F}(x, y) = \left(\frac{y}{x^2y^2 + 1}, \frac{x}{x^2y^2 + 1}\right).$$

Is $\varphi(x, y)$ the potential of $\mathbf{F}$?

Conservative vector field

If there is a function $\varphi : \mathbb{R}^3 \mapsto \mathbb{R}$ such that $\nabla\varphi = \mathbf{F}$, then $\mathbf{F}$ is a conservative vector field. φ is a potential for $\mathbf{F}$.

Example

Let $\varphi(x, y) = \arctan(xy)$. Let

$$\mathbf{F}(x, y) = \left(\frac{y}{x^2y^2 + 1}, \frac{x}{x^2y^2 + 1} \right).$$

Is $\varphi(x, y)$ the potential of $\mathbf{F}$?

Answer: Yes.

The gravitational filed is conservative

Let

$$\varphi(x, y, z) = \frac{km}{|(x, y, z)|}.$$

Then

$$\nabla\varphi(x, y, z) = \mathbf{F}(x, y, z) = -\frac{km}{|(x, y, z)|^3}(x, y, z).$$

Thus the vector field $\mathbf{F}$ is conservative.

Necessary condition for conservative field

If a vector field $\mathbf{F}(x, y, z) = (P, Q, R)$ is conservative, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

Necessary condition for conservative field

If a vector field $\mathbf{F}(x, y, z) = (P, Q, R)$ is conservative, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

If a planar vector field $\mathbf{F}(x, y) = (P, Q)$ is conservative, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Necessary condition for conservative field

If a vector field $\mathbf{F}(x, y, z) = (P, Q, R)$ is conservative, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

If a planar vector field $\mathbf{F}(x, y) = (P, Q)$ is conservative, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Example

Show that the velocity field $\mathbf{v}(x, y) = (-\Omega y, \Omega x)$ is not conservative.

Example

Is the following vector field conservative?

$$\mathbf{F}(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right)$$

Answer: No.

Example – Finding the potential function

What is potential function of the vector field $\mathbf{F}(x, y) = (x, -y)$?

Example – Finding the potential function

What is potential function of the vector field $\mathbf{F}(x,y) = (x, -y)$?

Answer: The potential function is of the form

$$\varphi(x,y) = \frac{x^2 - y^2}{2} + C_2$$

since

$$\nabla\varphi(x,y) = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial x}\right) = (x, -y) = \mathbf{F}.$$

Quiz

Find the potential function of the conservative field

$$\mathbf{F}(x, y, z) = (x, -2y, 3z).$$

Quiz

Find the potential function of the conservative field

$$\mathbf{F}(x, y, z) = (x, -2y, 3z).$$

Hint: The potential function φ satisfies

$$\left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}\right) = (x, -2y, 3z)$$

Quiz

Find the potential function of the conservative field

$$\mathbf{F}(x, y, z) = (x, -2y, 3z).$$

Hint: The potential function φ satisfies

$$\left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}\right) = (x, -2y, 3z)$$

Thus

$$\varphi(x, y, z) = \int x \mathrm{d}x = \frac{x^2}{2} + C_2(y, z).$$

Quiz

Find the potential function of the conservative field

$$\mathbf{F}(x, y, z) = (x, -2y, 3z).$$

Hint: The potential function φ satisfies

$$\left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right) = (x, -2y, 3z)$$

Thus

$$\varphi(x, y, z) = \int x\mathrm{d}x = \frac{x^2}{2} + C_2(y, z).$$

Then use

$$\frac{\partial\varphi}{\partial y} = \frac{\partial C_2(y, z)}{\partial y} = -2y$$

to get

$$C_2(y, z) = \int -2y\mathrm{d}y = -y^2 + C_1(z)$$

Quiz

Find the potential function of the conservative field

$$\mathbf{F}(x, y, z) = (x, -2y, 3z).$$

Hint: The potential function φ satisfies

$$\left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}\right) = (x, -2y, 3z)$$

Thus

$$\varphi(x, y, z) = \int x \mathrm{d}x = \frac{x^2}{2} + C_2(y, z).$$

Then use

$$\frac{\partial \varphi}{\partial y} = \frac{\partial C_2(y, z)}{\partial y} = -2y$$

to get

$$C_2(y, z) = \int -2y \mathrm{d}y = -y^2 + C_1(z)$$

What is $C_1(z)$??

Equipotential surfaces and equipotential curves

Equipotential surfaces

If $\varphi(x, y, z)$ is the potential function of $\mathbf{F}(x, y, z)$, then the level surfaces $\varphi(x, y, z) = C$ are called equipotential surfaces of $\mathbf{F}$.

Equipotential surfaces

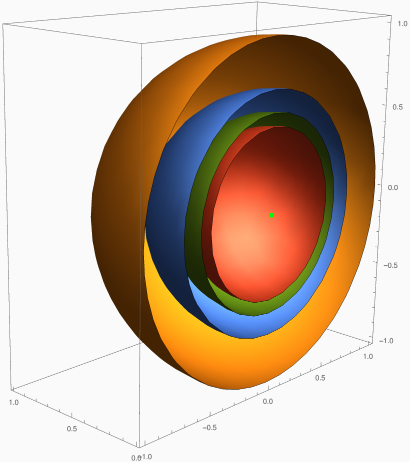
If $\varphi(x, y, z)$ is the potential function of $\mathbf{F}(x, y, z)$, then the level surfaces $\varphi(x, y, z) = C$ are called equipotential surfaces of $\mathbf{F}$.

Example

Since the potential function of the gravitational field is

$$\varphi(x, y, z) = \frac{km}{|(x, y, z)|},$$

the equipotential surfaces are spheres with center at $(0, 0, 0)$.



Equipotential curves

If $\varphi(x, y)$ is a potential function of $\mathbf{F}(x, y)$, then the level curves $\varphi(x, y) = C$ are called equipotential curves of $\mathbf{F}$.

Equipotential curves

If $\varphi(x, y)$ is a potential function of $\mathbf{F}(x, y)$, then the level curves $\varphi(x, y) = C$ are called equipotential curves of $\mathbf{F}$.

Example

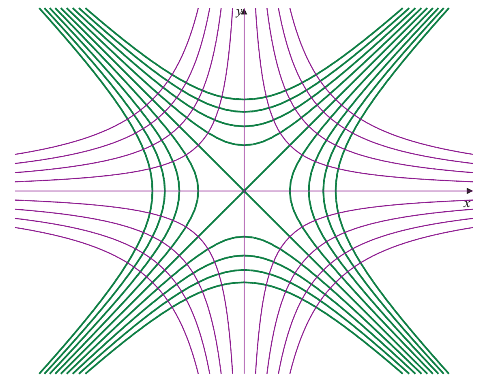
Since a potential function of $\mathbf{F}(x, y) = (x, -y)$ is

$$\varphi(x, y) = \frac{x^2 - y^2}{2}$$

the equipotential curves are of the form $\frac{x^2-y^2}{2} = C$.

Equipotential curves

If $\varphi(x, y)$ is a potential function of $\mathbf{F}(x, y)$, then the level curves $\varphi(x, y) = C$ are called equipotential curves of $\mathbf{F}$.



The equipotential curves (violet) and field lines (green) of $\mathbf{F} = (x, -y)$

😲Equipotential curves and field lines intersect at right angel.