Lecture 18 – Line Integral

Several Variable Calculus, 1MA017

Xing Shi Cai

Autumn 2019

Department of Mathematics, Uppsala University, Sweden

Summary

Please watch this video before the lecture: 19

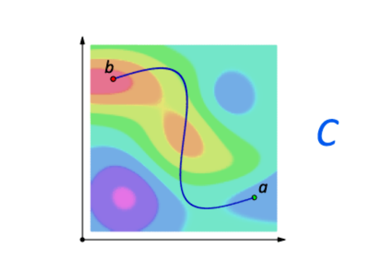
Today we will talk about

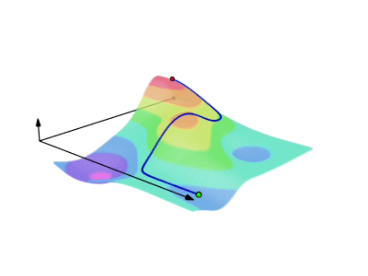
- 15.3 Line Integrals
- 15.4 Line Integrals of Vector Fields

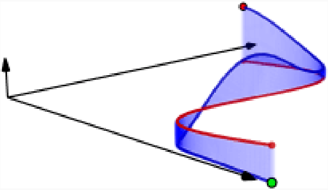
Line integrals

The problem

We are given a function $f(x,y)$ and a smooth curve $\mathcal{C}$. How can we compute the area under $f(x,y)$ along this curve $\mathcal{C}$?



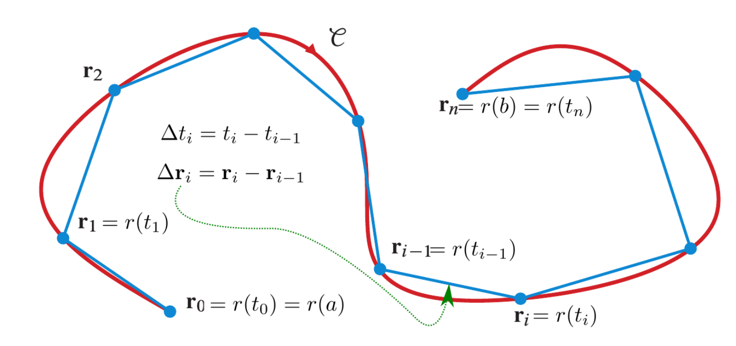




Review: The arc-length of a curve

Then the length of the curve $\mathcal{C}$ is approximately

$$\sum_{i=1}^{n} |\Delta \mathbf{r}_i|.$$



A curve with parametrization $r(t), t \in [a, b]$

Review: The arc-length of a curve

Then the length of the curve $\mathcal{C}$ is approximately

$$\sum_{i=1}^{n} |\Delta \mathbf{r}_i|.$$

Thus the length of the arc is defined as the

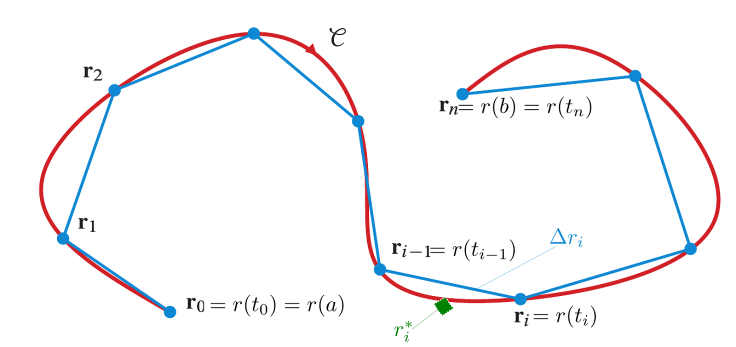
$$\lim \sum_{i=1}^{n} |\Delta \mathbf{r}_i| = \lim \sum_{i=1}^{n} \left|\frac{\Delta \mathbf{r}_i}{\Delta t_i}\right| \Delta t_i = \int_a^b |\mathbf{r}'(t)| \mathrm{d}t$$

where the limit is taken as $\max \Delta t_i \to 0$.

Line integral

Similarly, the area under f along the curve is approximately

$$\sum_{i=1}^{n} f(\mathbf{r}_i^*)\left|\Delta \mathbf{r}_i\right|$$



Line integral

Similarly, the area under f along the curve is approximately

$$\sum_{i=1}^{n} f(\mathbf{r}_i^*)\big|\Delta\mathbf{r}_i\big|$$

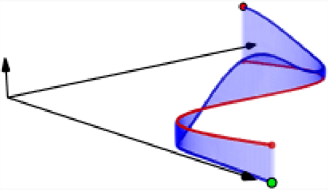
Thus we define the line integral of f along $\mathcal{C}$ as

$$\begin{aligned}\int_{\mathcal{C}} f\mathrm{d}s &= \lim \sum_{i=1}^{n} f(\mathbf{r}_i^*)\big|\Delta\mathbf{r}_i\big| \\ &= \lim \sum_{i=1}^{n} f(\mathbf{r}_i^*)\left|\frac{\Delta\mathbf{r}_i}{\Delta t_i}\right|\Delta t_i \\ &= \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\mathrm{d}t,\end{aligned}$$

as $\max \Delta t_i \to 0$.

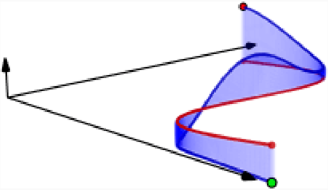
Interpretation of line integral

The line integral $\int_{\mathcal{C}} f \mathrm{d}s$ can be seen as the area under f along $\mathcal{C}$.



Interpretation of line integral

The line integral $\int_{\mathcal{C}} f \mathrm{d}s$ can be seen as the area under f along $\mathcal{C}$.

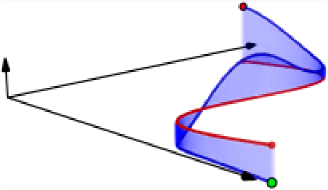


If $f(\mathbf{r}(t)) = 1$, then

$$\int_{\mathcal{C}} f \mathrm{d}s = \int_{\mathcal{C}} 1 \mathrm{d}s = \text{the length of } \mathcal{C}.$$

Interpretation of line integral

The line integral $\int_{\mathcal{C}} f \mathrm{d}s$ can be seen as the area under f along $\mathcal{C}$.



If $f(\mathbf{r}(t)) = 1$, then

$$\int_{\mathcal{C}} f \mathrm{d}s = \int_{\mathcal{C}} 1 \mathrm{d}s = \text{the length of } \mathcal{C}.$$

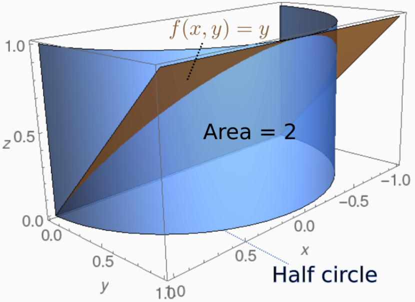
If $f(\mathbf{r}(t))$ is the density of the curve, then

$$\int_{\mathcal{C}} f \mathrm{d}s = \text{the mass of } \mathcal{C}.$$

Example – circle

Let $\mathcal{C}$ be the upper half of the unit circle. Show that

$$I = \int_{\mathcal{C}} y \mathrm{d}s = 2$$



Example – circle

Let $\mathcal{C}$ be the upper half of the unit circle. Show that

$$I = \int_{\mathcal{C}} y \mathrm{d}s = 2$$

Solution: We can parametrize $\mathcal{C}$ by $\mathbf{r}(t) = (\cos(t), \sin(t)), 0 \leq t \leq \pi$. Then

$$|\mathbf{r}'(t)| = |(-\sin(t), \cos(t))| = 1$$

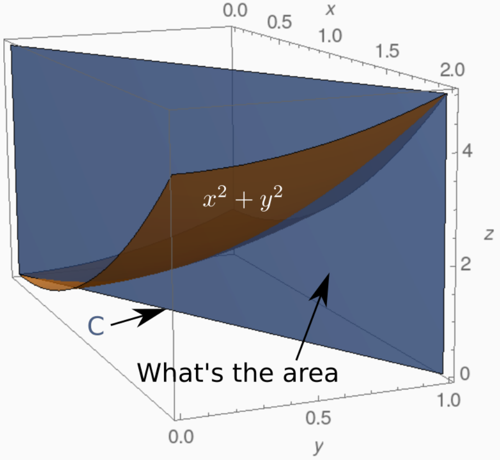
Thus

$$I = \int_0^\pi f(\mathbf{r}(t))\mathbf{r}'(t)\mathrm{d}t = \int_0^\pi \cos(t) \times 1 \mathrm{d}t = 2.$$

Example – Straight line

Let $\mathcal{C}$ be the line from the origin to $(2, 1)$. Compute

$$I = \int_{\mathcal{C}} (x^2 + y^2) \mathrm{d}s$$



Example – Straight line

Let $\mathcal{C}$ be the line from the origin to $(2,1)$. Compute

$$I = \int_{\mathcal{C}} (x^2 + y^2) \mathrm{d}s$$

Solution: We can parametrize $\mathcal{C}$ by $\mathbf{r}(t) = (2t, t), t \ \in [0,1]$. Then

$$|\mathbf{r}'(t)| = |(2,1)| = \sqrt{5}$$

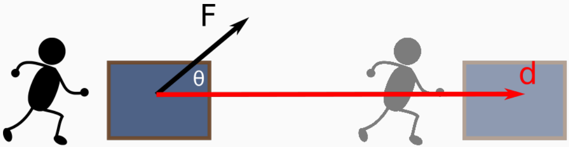
Thus

$$I = \int_0^{\pi} f(\mathbf{r}(t))\mathbf{r}'(t)\mathrm{d}t = \int_0^1 (4t^2 + t^2)\sqrt{5}\mathrm{d}t = \frac{5\sqrt{5}}{3}.$$

Line integrals of vector fields

What is work in physics?

In physics, the work done by a constant force $\mathbf{F}$ in moving an object along a straight line $\mathbf{d}$ is the dot product $W = \mathbf{F} \bullet \mathbf{d}$.



$$W = F \cdot d = |F||d|\cos(\theta)$$

What is work in physics?

In physics, the work done by a constant force $\mathbf{F}$ in moving an object along a straight line $\mathbf{d}$ is the dot product $W = \mathbf{F} \bullet \mathbf{d}$.

What if the force $\mathbf{F}$ is not a constant and the move is along a curve?

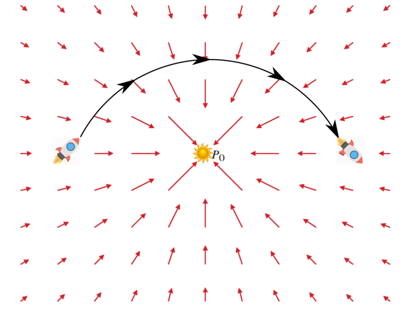
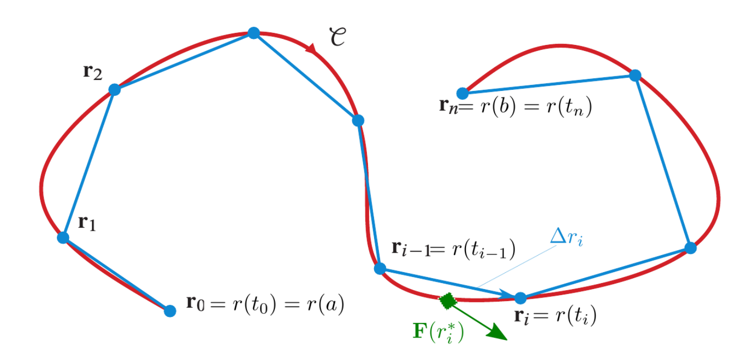


Figure 15.1 The gravitational field of a point mass located at P_0

Work in a vector field

Along the smooth curve $\mathcal{C}$, the work done by a vector field $\mathbf{F}$ can be approximated by

$$\sum_{i=1}^{n} \mathbf{F}(\mathbf{r}_i^*) \bullet \Delta\mathbf{r}_i.$$



This is the line integral of $\mathbf{F}$ along $\mathcal{C}$.

Work in a vector field

Along the smooth curve $\mathcal{C}$, the work done by a vector field $\mathbf{F}$ can be approximated by

$$\sum_{i=1}^{n} \mathbf{F}(\mathbf{r}_i^*) \bullet \Delta\mathbf{r}_i.$$

Thus we define

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F} \bullet \mathrm{d}\mathbf{r} &= \lim \sum_{i=1}^{n} \mathbf{F}(\mathbf{r}_i^*) \bullet \Delta\mathbf{r}_i \\ &= \lim \sum_{i=1}^{n} \left(\mathbf{F}(\mathbf{r}_i^*) \bullet \frac{\Delta\mathbf{r}_i}{\Delta t_i} \Delta t_i \right) \\ &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t) \mathrm{d}t\end{aligned}$$

This is the line integral of $\mathbf{F}$ along $\mathcal{C}$.

15.4 Example 1

Let $\mathbf{F}(x, y) = (y^2, 2xy)$. Compute the line integral

$$\int_{\mathcal{C}} \mathbf{F} \bullet \mathrm{d}r$$

from $(0, 0)$ to $(1, 1)$ along the three curves shown in the picture.

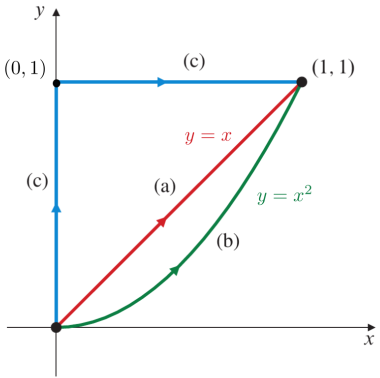


Figure 15.9 Three paths from (0, 0) to

Line integrals of conservative vector fields

Let $\mathbf{F}$ be a conservative vector field with potential function φ. Let $\mathcal{C}$ be a smooth curve starts at (x_0, y_0) and ends at (x_1, y_1). Then

$$\int_{\mathcal{C}} \mathbf{F} \bullet \mathrm{d}\mathbf{r} = \varphi(x_1, y_1) - \varphi(x_0, y_0)$$

Line integrals of conservative vector fields

Let $\mathbf{F}$ be a conservative vector field with potential function φ. Let $\mathcal{C}$ be a smooth curve starts at (x_0, y_0) and ends at (x_1, y_1). Then

$$\int_{\mathcal{C}} \mathbf{F} \bullet \mathrm{d}\mathbf{r} = \varphi(x_1, y_1) - \varphi(x_0, y_0)$$

Proof Let $\mathbf{r}(t), t \in [a, b]$ be a parametrization of $\mathcal{C}$. Let $g(t) = \varphi(\mathbf{r}(t))$. Then

Line integrals of conservative vector fields

Let $\mathbf{F}$ be a conservative vector field with potential function φ. Let $\mathcal{C}$ be a smooth curve starts at (x_0, y_0) and ends at (x_1, y_1). Then

$$\int_{\mathcal{C}} \mathbf{F} \bullet \mathrm{d}\mathbf{r} = \varphi(x_1, y_1) - \varphi(x_0, y_0)$$

Proof Let $\mathbf{r}(t), t \in [a, b]$ be a parametrization of $\mathcal{C}$. Let $g(t) = \varphi(\mathbf{r}(t))$. Then

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F} \bullet \mathrm{d}\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\mathrm{d}t \\ &= \int_a^b \varphi'(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\mathrm{d}t = \int_a^b g'(t)\mathrm{d}t\end{aligned}$$

Line integrals of conservative vector fields

Let $\mathbf{F}$ be a conservative vector field with potential function φ. Let $\mathcal{C}$ be a smooth curve starts at (x_0, y_0) and ends at (x_1, y_1). Then

$$\int_{\mathcal{C}} \mathbf{F} \bullet \mathrm{d}\mathbf{r} = \varphi(x_1, y_1) - \varphi(x_0, y_0)$$

Proof Let $\mathbf{r}(t), t \in [a, b]$ be a parametrization of $\mathcal{C}$. Let $g(t) = \varphi(\mathbf{r}(t))$. Then

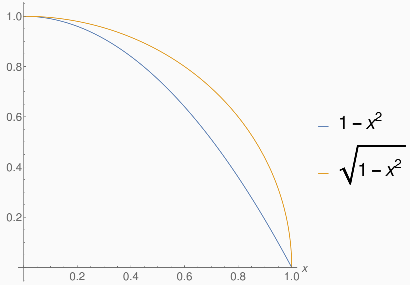
$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \bullet \mathrm{d}\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\mathrm{d}t \\
&= \int_a^b \varphi'(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\mathrm{d}t = \int_a^b g'(t)\mathrm{d}t \\
&= g(b) - g(a) = \varphi(\mathbf{r}(b)) - \varphi(\mathbf{r}(a)) \\
&= \varphi(x_1, y_1) - \varphi(x_0, y_0)
\end{aligned}$$

Example – When the vector field is not conservative.

Let $\mathbf{F}(x, y) = (-y, x)$. Compute the line integral

$$\int_{\mathcal{C}} \mathbf{F} \bullet \mathrm{d}r$$

from $(1, 0)$ to $(0, 1)$ along the two curves shown in the picture.

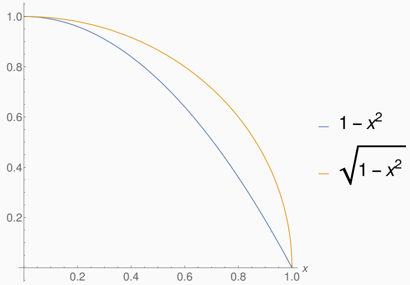


Example – When the vector field is not conservative.

Let $\mathbf{F}(x, y) = (-y, x)$. Compute the line integral

$$\int_{\mathcal{C}} \mathbf{F} \bullet \mathrm{d}r$$

from $(1, 0)$ to $(0, 1)$ along the two curves shown in the picture.



Answer: $4/3$ and $\pi/2$.

Possible exam problem

Consider the planar vector field

$$\mathbf{F}(x, y) = (2x + 2y, 2x + 2y)$$

1. Find all potential functions of $\mathbf{F}$.
2. Compute the integral $\int_{\mathcal{C}} \mathbf{F} \bullet \mathrm{d}\mathbf{r}$ along any curve starts at $(0, 0)$ and ends at $(1, 1)$.

Possible exam problem

Consider the planar vector field

$$\mathbf{F}(x, y) = (2x + 2y, 2x + 2y)$$

1. Find all potential functions of $\mathbf{F}$.
2. Compute the integral $\int_{\mathcal{C}} \mathbf{F} \bullet \mathrm{d}\mathbf{r}$ along any curve starts at $(0, 0)$ and ends at $(1, 1)$.

Answer:

1. $\varphi(x, y) = x^2 + 2xy + y^2 + C_1$
2. $\varphi(1, 1) - \varphi(0, 0) = 4$

Let D be an open, connected domain, and let F be a smooth vector field on D. Then the following are equivalent:

1. $\mathbf{F}$ is conservative.
2. $\oint_{\mathcal{C}} \mathbf{F} \bullet \mathrm{d}\mathbf{r} = 0$ for any smooth, closed curve $\mathcal{C}$.
3. All line integrals of $\mathbf{F}$ with the same start and end are independent of the path.

Quiz

Let $\mathbf{F}(x, y) = (ye^{xy}, xe^{xy})$.

1. Determine if $\mathbf{F}$ is conservative.
2. Compute the integral $\int_{\mathcal{C}} \mathbf{F} \bullet \mathrm{d}\mathbf{r}$ along:
   a) $\mathbf{r}(t) = (te^t, e^{t-1})$, $0 \leq t \leq 1$.
   b) The line from $(e, 1)$ to $(0, e^{-1})$.