Lecture 19 – Green's Theorem in the Plane

Several Variable Calculus, 1MA017

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Summary

Please watch this video before the lecture: 20

Today we will talk about

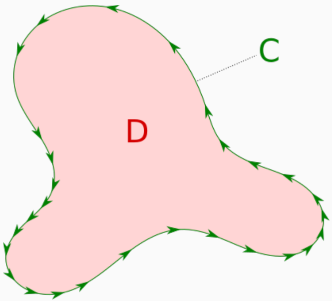
- 16.3 Green's Theorem in the Plane

Green's Theorem in the Plane –Theorem 6, 16.3

Let D be a region with a boundary curve $\mathcal{C}$ that is oriented counter clockwise. Let $\mathbf{F}(x, y) = (P, Q)$ be a vector field. Then

$$\oint_{\mathcal{C}} \mathbf{F} \bullet \mathrm{d}\mathbf{r} = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A.$$

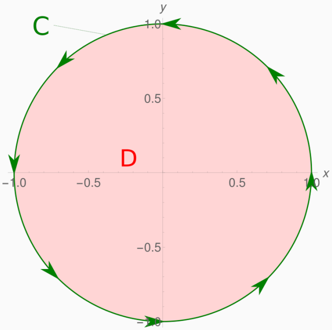
Note When $\mathbf{F}$ is conservative, both sides are 0.



Examples

Let $\mathcal{C}$ be the unit circle, counter-clockwise oriented, and $\mathbf{F}(x, y) = (P, Q) = (-y, x)$. Then by Green's theorem

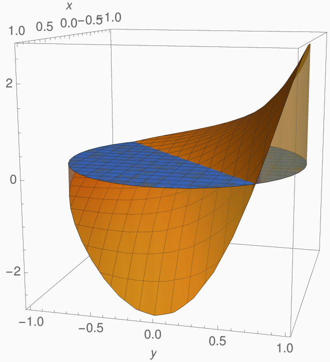
$$\oint_{\mathcal{C}} \mathbf{F} \bullet \mathrm{d}\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A = \iint_D 2 \, \mathrm{d}A = 2\pi.$$



Example

Let $\mathcal{C}$ be the unit circle, counter-clockwise oriented, and $\mathbf{F}(x, y) = (P, Q) = (e^{xy}, e^{xy})$. Then by Green's theorem

$$\oint_{\mathcal{C}} \mathbf{F} \bullet \mathrm{d}\mathbf{r} = \iint_{D} (ye^{xy} - xe^{xy}) \,\mathrm{d}A = 0$$



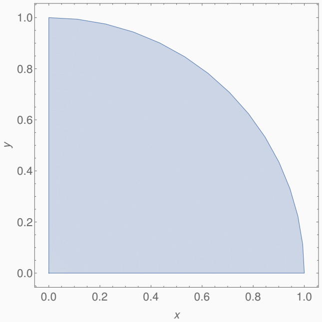
Example

Let C be counter-clockwise oriented boundary of the quarter-disk $D : 0 \leq x^2 + y^2 \leq 1, x \geq 0, y \geq 0$.

Let $\mathbf{F}(x, y) = (P, Q) = (x - y^3, y^3 + x^3)$.

Then by Green's theorem

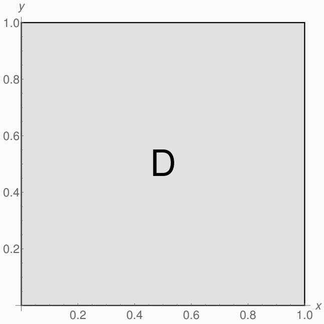
$$\oint_{\mathcal{C}} \mathbf{F} \bullet \mathrm{d}\mathbf{r} = \iint_D 3(x^2 - (-y^2))\, \mathrm{d}A = \frac{3}{8}\pi$$



Quiz

Let $\mathcal{C}$ is the unit square, counter-clockwise oriented. Let $\mathbf{F}(x, y) = (P, Q) = (x^2 - y^2, 2xy)$. Then by Green's theorem

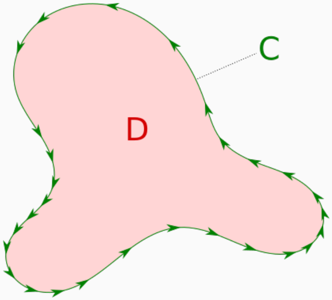
$$\oint_{\mathcal{C}} \mathbf{F} \bullet \mathrm{d}\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A = ??.$$



Compute area with Green's theorem

Let $\mathbf{F}(x, y) = (P, Q) = (-y/2, x/2)$. Let D be a domain and $\mathcal{C}$ be its boundary. Then by Green's Theorem,

$$\text{area of } D = \iint_D 1 \mathrm{d}A = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A = \oint_{\mathcal{C}} \mathbf{F} \bullet \mathrm{d}\mathbf{r}.$$



Example

Let $\mathcal{C}$ be the closed curve given by

$$\mathbf{r}(t) = (3\cos t + 3\sin t, 2\sin t - 2\cos t), \qquad 0 \leq t < 2\pi.$$

Compute the area enclosed in $\mathcal{C}$.

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Compute the area enclosed in $\mathcal{C}$.

Answer: By Green's Theorem, the area bounded by $\mathcal{C}$ is given by

$$\oint_{\mathcal{C}} \mathbf{F} \bullet \mathrm{d}\mathbf{r} = 12\pi$$

where $\mathbf{F}(x, y) = (P, Q) = (-y/2, x/2)$.

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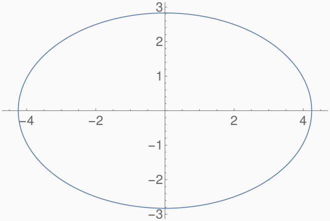
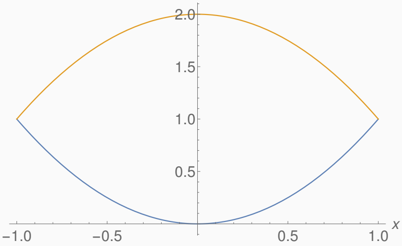


Quiz

Calculate the area of the area between $y = x^2$ and $y = 2 - x^2$ in the interval $-1 \leq x \leq 1$ using Green's formula.



Hint Then the area bounded by $\mathcal{C}$ is given by

$$\oint_{\mathcal{C}} \mathbf{F} \bullet \mathrm{d}\mathbf{r}$$

where $\mathbf{F}(x, y) = (P, Q) = (-y/2, x/2)$.

The end!