Midterm exam

1MA017 Several variable calculus, limited version, Autumn 2019

Question 1

1. Consider the curve $\mathbf{r}(t) = \left(\cos\left(\frac{t}{\sqrt{2}}\right), \sin\left(\frac{t}{\sqrt{2}}\right), \frac{t}{\sqrt{2}}\right)$ in $\mathbb{R}^3$ for $t \in [0, 2\sqrt{2}\pi]$.

(a) Compute the velocity and the acceleration of the curve.

(b) Compute the length of the curve.

(c) Draw the curve.

(a)

The velocity is

$$r'(t) = (-\frac{\sin\left(\frac{t}{\sqrt{2}}\right)}{\sqrt{2}}, \frac{\cos\left(\frac{t}{\sqrt{2}}\right)}{\sqrt{2}}, \frac{1}{\sqrt{2}})$$

The acceleration is

$$r''(t) = (-\frac{1}{2}\cos\left(\frac{t}{\sqrt{2}}\right), -\frac{1}{2}\sin\left(\frac{t}{\sqrt{2}}\right), 0)$$

(b)

Since

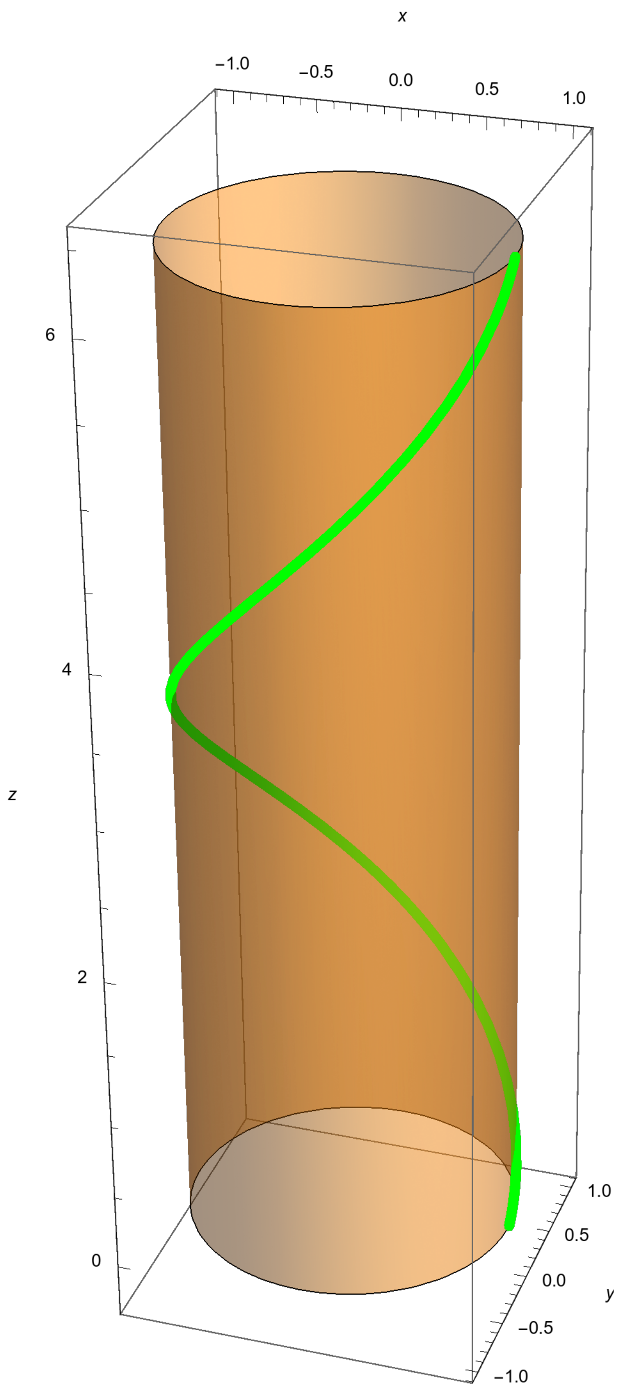
$$|r'(t)| = \sqrt{\frac{1}{2}\left|\sin\left(\frac{t}{\sqrt{2}}\right)\right|^2 + \frac{1}{2}\left|\cos\left(\frac{t}{\sqrt{2}}\right)\right|^2 + \frac{1}{2}} = 1$$

The length of the curve is

$$\int_0^{2\sqrt{2}\pi} |r'(t)|\, dt = 2\sqrt{2}\,\pi$$

(c)

See the green curve below. (The cylinder is just to make the picture easier to interpret. You don’t have to draw it.)



Question 2

2. Let $f(x,y) = \ln(x+y^2)$.

(a) Compute $p_2(x,y)$, the degree 2 Taylor polynomial of $f(x,y)$, at the point $(1,0)$.

(b) Use the result of (b) to compute an approximation of $f(1.01,-0.02)$.

(a)

We have

$$f_1(x, y) = \frac{1}{x+y^2}, \quad f_2(x, y) = \frac{2y}{x+y^2}$$

and

$$f_{11}(x, y) = -\frac{1}{(x+y^2)^2}, \; f_{12}(x, y) = f_{21}(x, y) = -\frac{2y}{(x+y^2)^2}, \; f_{22}(x, y) = \frac{2(x-y^2)}{(x+y^2)^2}$$

So we have

$$\begin{aligned}
&p_2(x, y) \\
&= f(1,0) + f_1(1,0)(x-1) + f_2(1,0)(y-0) + \frac{1}{2}\left(f_{11}(1,0)(x-1)^2 + 2 f_{12}(1,0)(x-1)(y-0) + f_{22}(1,0)(y-0)\right)^2 \\
&= (x-1) + \frac{1}{2}\left(-(x-1)^2 + 2y^2\right) \\
&= (x-1) - \frac{1}{2}(x-1)^2 + y^2 \\
&= -\frac{x^2}{2} + 2x + y^2 - \frac{3}{2}
\end{aligned}$$

(b)

We have

$p_2(1.01, -0.02) = 0.01035$

(Actually, $f(1.01, -0.02) = 0.0103463$, so the approximation is quite accurate).

Question 3

3. Consider the function $f(x,y) = x^2 + xy + 2y^2$.

(a) Find the critical points of $f(x,y)$.

(b) Identify the type each of the critical points, i.e., are they local maxima, local minima or saddle points?

(a)

Since

$f_1(x, y) = 2x + y,\ f_2(x, y) = x + 4y$

Solving the equations

$2x + y = 0$
$x + 4y = 0$

shows that the only critical point is

(0, 0)

(b)

Since

$f(x, y) = x^2 + xy + 2y^2 = \left(x + \frac{y}{2}\right)^2 + \frac{7y^2}{4} \geq 0 = f(0, 0)$

The point (0, 0) is a local minimal point. (It's also a absolute minimal point.)

Alternatively solution of (b)

We can compute the Hessian matrix of $f(x, y)$

$$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$$

Then we have $D_1 = 2$, $D_2 = 2 \times 4 - 1 \times 1 = 7$. Since $D_1 > 0$ and $D_2 > 0$, we also have (0, 0) is a local minimal point.

Question 4

4. Consider the function $f(x, y) = e^{5x}\sin(5y)$.

(a) Show that $f(x, y)$ satisfies the Laplace's equation, i.e.,

$$f_{11}(x, y) + f_{22}(x, y) = 0.$$

(b) Let $g(t) = f(\cos(t), \sin(t))$. Compute $g'(t)$.

(a)

We can compute the second order partial derivatives

$f_{11}(x, y) = 25\, e^{5x} \sin(5y), \ f_{22}(x, y) = -25\, e^{5x} \sin(5y)$

There fore

$f_{11}(x, y) + f_{22}(x, y) = 0$

(b)

We have

$g(t) = \sin(5 \sin(t))\, e^{5 \cos(t)}$

This is just a one-variable function. So taking derivative with respect to t, we get

$5\, e^{5 \cos(t)} \cos(t + 5 \sin(t))$

Alternative solution of (b)

We can use the chain rule. Since

$f_1(x, y) = 5\, e^{5x} \sin(5y), \ f_2(x, y) = 5\, e^{5x} \cos(5y)$

Then

$g'(t)$

$= \cos(t)' f_1(x(t), y(t)) + \sin(t)' f_2(x(t), y(t))$

$= -\sin(t)\, 5\, e^{5x} \sin(5y) + \cos(t)\, 5\, e^{5x} \cos(5y)$

$= -\sin(t)\, 5\, e^{5\cos(t)} \sin(5 \sin(t)) + \cos(t)\, 5\, e^{5\cos(t)} \cos(5 \sin(t))$

$= 5\, e^{5\cos(t)} (-\sin(t) \sin(5 \sin(t)) + \cos(t) \cos(5 \sin(t)))$

$= 5\, e^{5\cos(t)} \cos(t + 5 \sin(t))$