Assignment 1 Solution

Section 2.9

Exercise 10

By the first condition, there are 10×9 options for the first and the last symbols.

For the second condition, we have $\binom{13}{4}$ choices for put 4 ‘t’ in the 13 positions left.

For the 3rd condition, we have $\binom{9}{3}$ choices for the positions.

Once these positions are decided, there $5 \times 4 \times 3$ choices to put 3 distinct letters from {a, e, i, o, u} there. That is $5 \times 4 \times 3 \times \binom{9}{3}$ choices in total.

Now we have 6 positions left to fill, we cannot use ‘t’ or {a, e, i, o, u} again. So there are $26 + 10 - 1 - 5 = 30$ choices for each of these 6 positions. So in total we have

$$10 \times 9 \times \binom{13}{4} \times 5 \times 4 \times 3 \times \binom{9}{3} \times 30^6 \tag{1}$$

which is

Out[204]= 236 432 196 000 000 000

You do not need to actually compute this. Getting a formula like (1) is good.

Exercise 26

There are in total $\binom{14+73}{14} = \binom{87}{14}$ paths from (0, 0) to (14, 73).

There are $\binom{6+37}{6} = \binom{43}{6}$ paths from (0, 0) to (6, 37)

From (6, 37) to (14, 73) there are

$$\binom{(14-6)+(73-37)}{14-6} = \binom{44}{8}$$

paths.

So the number of path from (0, 0) to (14, 73) going through (6, 37) is

$$\binom{43}{6}\binom{44}{8}$$

So the number of path from (0, 0) to (14, 73) which does **not** go through (6, 37) is

$$\binom{87}{14} - \binom{43}{6}\binom{44}{8} \tag{2}$$

which is

Out[205]= 4 328 217 105 260 492

You do not need to actually compute this. Getting a formula like (2) is good enough.

Exercise 33

In general, you can prove these things by finding a bijection with Dyck path or lattice path.

You do not need to formally justify your bijection.

Finding a correct bijection will be enough to get full marks in the exam.

Though you should definite **check** for $n = 3$ or $n = 4$ that your bijection gives all the Dyck path of length $2n$.

(a)

There are many ways to do this. For example, check this one.

Another way to do is to convert each left ‘(‘ to a +1 and each ‘a’ except the last one into -1.

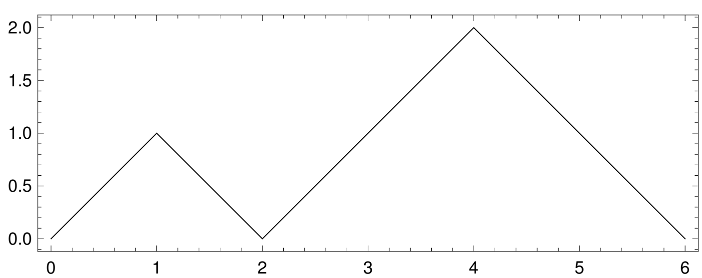
For example $(a_1((a_2\, a_3)\, a_4))$ corresponds to (1, -1, 1,1,-1,-1) . And by the result of (b), this corresponds to a Dyck path.

(b)

Think +1 as going up in a Dyck path and -1 as going down in a Dyck path.

For example, (1, -1, 1,1,-1,-1) corresponds to

Out[206]=



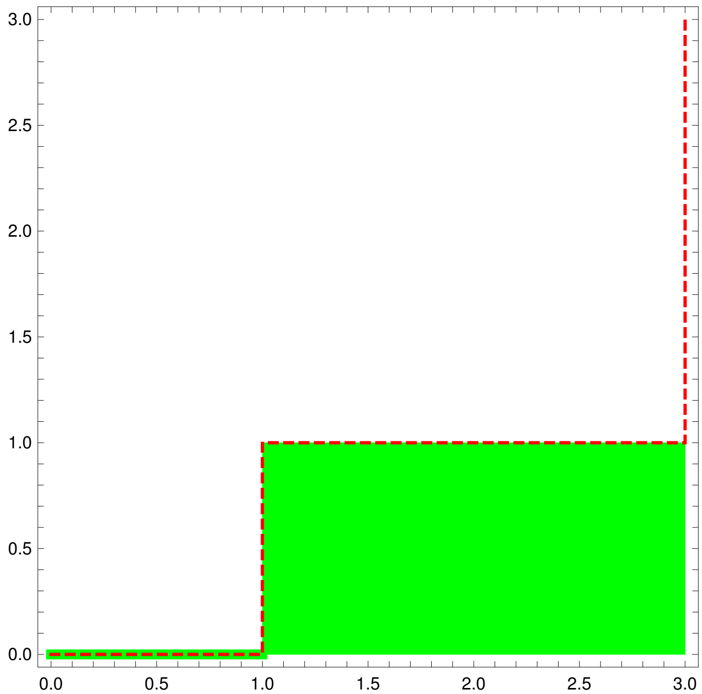
So there is a bijection between Dyck paths and such sequence.

(c)

One way to do this is to subtract 1 from each number, for example (1,2,2) becomes (0,1,1)

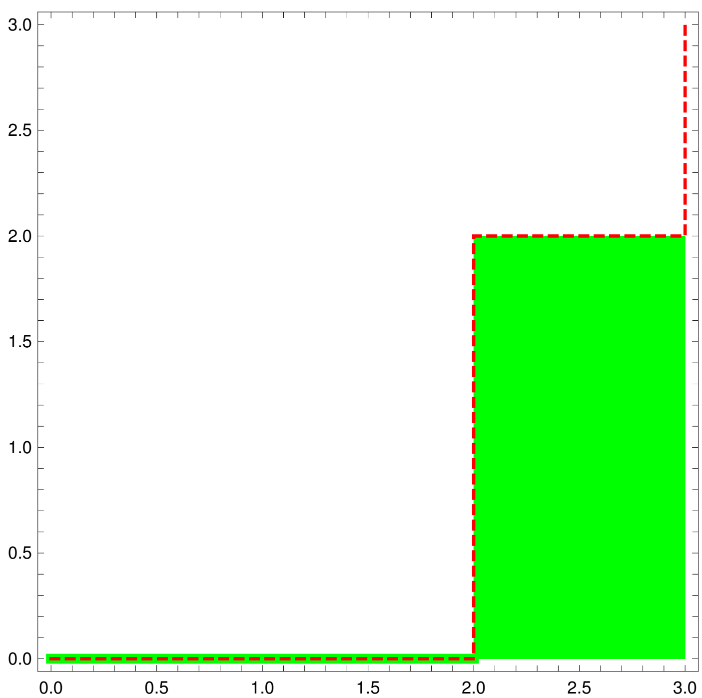
Then draw (0, 1,1) boxes in the first 3 column of a lattice grid, which gives a stair-step shape. The upper outline of these rectangles obviously give a **good** lattice path.

Out[237]=



Another example is (1, 1, 3). This becomes (0, 0, 2). And the picture is.

Out[236]=

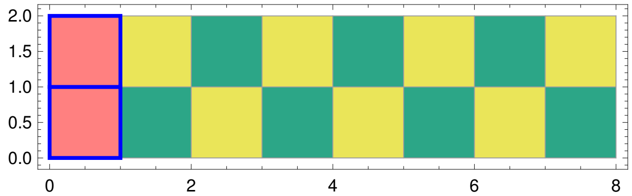


Section 3.11

Exercise 4

There 7 ways to start a tiling are the following ones.

Out[212]=



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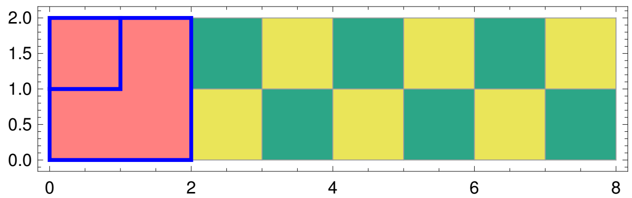

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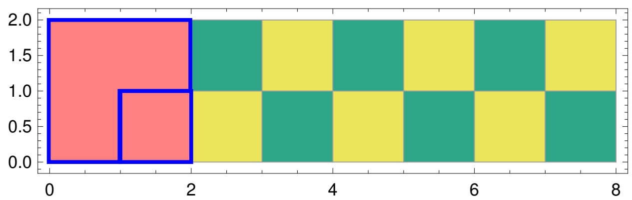

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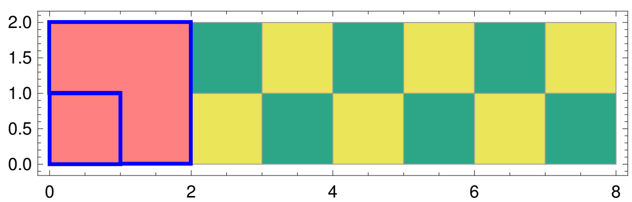

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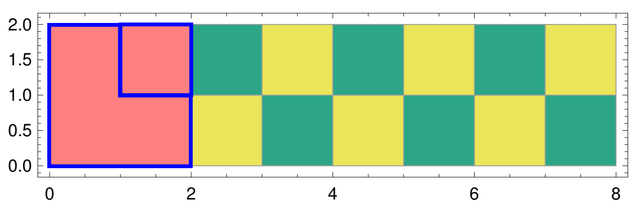
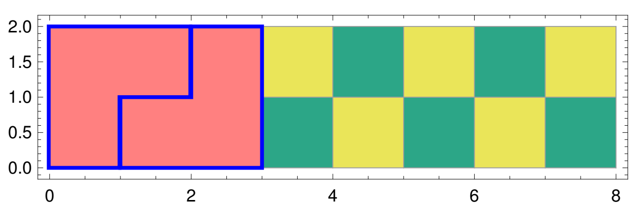
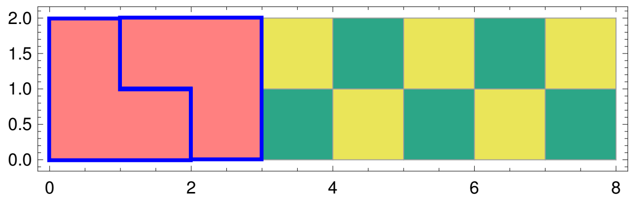


Out[218]=



Out[219]=



So the recursion we are seeking for is

$$t(n) = 2\,t(n-3) + 4\,t(n-2) + t(n-1)$$

We start with $t(0) = 1$ (there is only one way to do this, put nothing there), $t(1) = 1$ and $t(2) = 5$.

So $t(0)$ to $t(7)$ are.

Out[220]= {1, 1, 5, 11, 33, 87, 241, 655}

Exercise 18

$f(1)$ to $f(4)$ are

Out[221]= {1, 1, 2, 3}

The remainder of these numbers divided by 3 is

Out[222]= {1, 1, 2, 0}

So the statement is true for $n = 1 \ldots 4$. This is the base case.

Now we assume that the statement is true for all $n \le 4\,k$, where $k \ge 1$ is an integer.

Let $4\,k + 1 \le n \le 4\,(k+1)$. Then

$$\begin{aligned} &f(n) \\ &= f(n-2) + f(n-1) \\ &= \big(f(n-3) + f(n-4)\big) + f(n-2) + f(n-3) \\ &= \big(f(n-3) + f(n-4)\big) + \big(f(n-3) + f(n-4)\big) + f(n-3) \\ &= 3\,f(n-3) + 2\,f(n-4) \end{aligned}$$

So $f(n)$ divided by 3 has the same remainder as $f(n-4)$. By our induction hypothesis, $f(n)$ is a multiple of 3 if and only if $n = 4\,(k+1)$.

Exercise not from the textbook

If you try compute the sum for a few m and n, you can probably guess that

Out[224]//TraditionalForm=

$$\sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{n}{k}} = \frac{n+1}{-m+n+1}$$

Then you can prove this by induction on m.

For a proof without induction, see this page 173 of this book.