Assignment 2 -- Solutions

Section 2.9

16

(a)

By the balls and bars argument, this is $\binom{63-1}{5-1} = \binom{62}{4}$.

(c)

The question is equivalent to ask the number of solutions for

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 63, \text{ all } x_i \geq 0.$

So this is $\binom{63+6-1}{6-1} = \binom{68}{5}$.

(e) Solution 1

The most straightforward way to do this is to consider 10 cases $x_2 = 0, \ldots, 9$ and sum up the number of ways to divide $63 - x_2$ among other variables. This gives

$$\sum_{x_2=0}^{9} \binom{(63 - x_2) + 4 - 1}{4 - 1} == \sum_{x_2=0}^{9} \binom{66 - x_2}{3} = \sum_{k=57}^{66} \binom{k}{3}$$

We can further simplify this

$$\sum_{k=57}^{66} \binom{k}{3} = \sum_{k=0}^{66} \binom{k}{3} - \sum_{k=0}^{56} \binom{k}{3} = \binom{66+1}{3+1} - \binom{56+1}{3+1} = \binom{67}{4} - \binom{57}{4} \qquad (1)$$

(e) Solution 2

While the above is an Okay answer, there's a nicer way to do it.

The total number of solutions without the restriction on x_2 is $\binom{63+5-1}{5-1} = \binom{67}{4}$.

Among them, there are some "bad" solutions in which $x_2 \geq 10$. The number of these bad solutions is $\binom{(63-10)+5-1}{5-1} = \binom{57}{4}$.

So the answer is

$$\binom{67}{4} - \binom{57}{4} \tag{2}$$

(e) Solution 3

We can also do this by GF. The GF counting the solutions of this problem is

$$\frac{1}{(1-x)^4}\left(1+x+x^2+\ldots+x^9\right)$$
$$== \frac{1-x^{10}}{(1-x)^5}$$
$$== \left(1-x^{10}\right)\sum_{k=0}^{\infty}\binom{-5}{k}(-x)^k$$
$$== \sum_{k=0}^{\infty}\binom{-5}{k}(-x)^k - \sum_{k=0}^{\infty}\binom{-5}{k}(-x)^{k+10}$$

So the coefficient of x^{63} in this GF is

$$\binom{-5}{63}(-1)^{63} - \binom{-5}{53}(-1)^{63} \tag{3}$$

It's easy to check this equals (2).

Section 8.8

8.8.2

(c)

$$\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} (2\,x)^n = \frac{1}{1-2\,x}$$

(h)

$$\sum_{n=0}^{\infty} x^{n+3} = \frac{x^3}{1-x}$$

(k)

Out[12]= $$\frac{6}{1+x^2}$$

8.8.8

$$\left(x^2+x^3+\ldots\right)\left(1+x+x^2+x^3\right)\left(1+x^3+x^6+\ldots\right)\left(x+x^2+\ldots+x^6\right)$$

$$== \frac{x^2}{1-x}\frac{1-x^4}{1-x}\frac{1}{1-x^3}\frac{x\,(1-x^6)}{1-x}$$

$$== \frac{x^3\,(1+x)^2\,(1+x^2)\,(1-x+x^2)}{(1-x)^2}$$

The coefficient of x^2 is 0 and the coefficient of x^3 is 1. This simply because there is no way do this with two papers. And the only possible solution for 3 papers is to give Alice two and Dave one.

8.8.17

There are several GF that gives you the correct answer. Here's just one possible way.

Step 1 -- Get the GF

The GF for parton into odd parts is

$$\prod_{m=1}^{\infty} \frac{1}{1-x^{2m-1}}$$

But we also know that it's the same as the GF for partition into distinct parts

$$\prod_{m=1}^{\infty} (1+x^m)$$

Since we cannot have a part of the partition of size >10, it suffices to expand the GF

Out[53]= $(1+x)\,(1+x^2)\,(1+x^3)\,(1+x^4)\,(1+x^5)\,(1+x^6)\,(1+x^7)\,(1+x^8)\,(1+x^9)\,(1+x^{10})$

and find the coefficient to x^{10}. It's Okay if you stop here with the correct GF (there are other correct forms).

Step 2

Of course we can expand this completely, but you need computer to do this unless you are really tough.

To do it by hand, note that first, although

$$(1+x^5)\,(1+x^6)\,(1+x^7)\,(1+x^8)\,(1+x^9)\,(1+x^{10}) == 1+x^5+x^6+x^7+x^8+x^9+x^{10}+x^{11}+\ldots$$

with the … containing many many things, we can drop everything after x^{10}.

So we only need to expand

$$(1+x)\,(1+x^2)\,(1+x^3)\,(1+x^4)\,(1+x^5+x^6+x^7+x^8+x^9+x^{10})$$

The **idea** is that in each step of the expansion, we drop anything that cannot contribute to the coefficient of x^{10}.

Step 3

By the same argument, since

$$(1+x^4)\,(1+x^5+x^6+x^7+x^8+x^9+x^{10})$$

$$= 1 + x^4 + x^5 + x^6 + x^7 + x^8 + 2x^9 + 2x^{10} \ldots$$

we only need to expand

$$(1+x)\,(1+x^2)\,(1+x^3)\,(1+x^4+x^5+x^6+x^7+x^8+2x^9+2x^{10})$$

Step 4

Again, since

$$(1+x^3)\,(1+x^4+x^5+x^6+x^7+x^8+2x^9+2x^{10})$$
$$= 1+x^3+x^4+x^5+x^6+2x^7+2x^8+3x^9+3x^{10}+\ldots$$

we only need to expand

$$(1+x)\,(1+x^2)\,(1+x^3+x^4+x^5+x^6+2x^7+2x^8+3x^9+3x^{10})$$

But note that $x^3 + x^4 + x^5 + x^6$ in the last factor is of no use to us, so it suffices to expand

$$(1+x)\,(1+x^2)\,(1+2x^7+2x^8+3x^9+3x^{10})$$

Step 5

Since

$$(1+x^2)\,(1+2x^7+2x^8+3x^9+3x^{10})$$
$$= 1+x^2+2x^7+2x^8+5x^9+5x^{10}+\ldots$$

we only need to expand

$$(1+x)\,(1+x^2+2x^7+2x^8+5x^9+5x^{10}).$$

But note that $x^2 + 2x^7 + 2x^8$ in the last factor is of no use to us, so it suffices to expand

$$(1+x)\,(1+5x^9+5x^{10}) = 1+x+5x^9+10x^{10}+\ldots$$

So the coefficient of x^{10} in

$$(1+x)\,(1+x^2)\,(1+x^3)\,(1+x^4)\,(1+x^5)\,(1+x^6)\,(1+x^7)\,(1+x^8)\,(1+x^9)\,(1+x^{10})$$

is 10. This is the answer.

8.8.20

(a)

$$\sum_{n=0}^{\infty} \frac{5^n}{n!} x^n = \sum_{n=0}^{\infty} \frac{(5x)^n}{n!} = e^{5x}$$

(e)

$$\sum_{n=0}^{\infty} \frac{n}{n!} x^n = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^n = x \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^{n-1} = x \sum_{n=0}^{\infty} \frac{1}{n!} x^n = x\, e^x$$

(f)

$$\sum_{n=0}^{\infty} \frac{\frac{1}{n+1}}{n!} x^n == \sum_{n=0}^{\infty} \frac{1}{(n+1)!} x^n == \frac{1}{x} \sum_{n=0}^{\infty} \frac{1}{(n+1)!} x^{n+1}$$

$$== \frac{1}{x} \left(\sum_{n=-1}^{\infty} \frac{1}{(n+1)!} x^{n+1} - 1 \right) == \frac{1}{x} \left(\sum_{n=0}^{\infty} \frac{1}{n!} x^n - 1 \right) == \frac{1}{x} (e^x - 1)$$

8.8.26

The EGF for a vowel is

$$\sum_{n=1}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n - 1 == e^x - 1$$

The EGF for T is

$$\sum_{n=3}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n - \left(1 + x + \frac{x^2}{2!}\right) = e^x - 1 - x - \frac{x^2}{2}$$

The EGF for Z is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} == 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

The EGF for an even digit is

$$\sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} = \frac{e^x + e^{-x}}{2}$$

The EGF for an odd digit is

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1} = \frac{e^x - e^{-x}}{2}$$

There are 26 – 5 – 2 == 19 letters left without any restriction, each of them has the EGF e^x

Since there are 5 vowels, 5 even digits, 5 odd digits, the EGF for the strings that we are looking for is

$$(e^x - 1)^5 \left(e^x - 1 - x - \frac{x^2}{2}\right) \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right) \left(\frac{e^x + e^{-x}}{2}\right)^5 \left(\frac{e^x - e^{-x}}{2}\right)^5 (e^x)^{19}$$

The challenge problem

(a) and (b)

The GF for a_n is

$$\prod_{m=1}^{r} \frac{1}{1 - x^m}$$

For $r = 2$, this is

Out[76]= $$\frac{1}{(-1+x)^2\,(1+x)}$$

This equals

Out[77]= $$\frac{1}{2\,(-1+x)^2}-\frac{1}{4\,(-1+x)}+\frac{1}{4\,(1+x)}$$

So by Newton's binomial theorem, the coefficient of x^n, i.e., a_n is

Out[78]= $$\frac{1}{4}+\frac{(-1)^n}{4}+\frac{1+n}{2}$$

So a_0, a_1, a_2, a_3 for $r == 2$ is

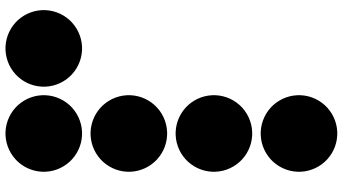
Out[79]= {1, 1, 2, 2}

This is easily verifiable without using GF.

(c) and (d)

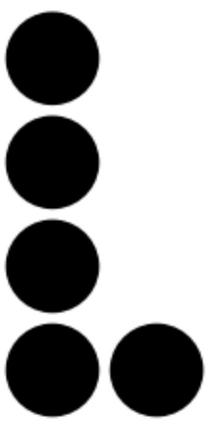
If we partition 5 = 2 + 1 + 1 + 1, we can picture this as

Out[127]=



If we flip this picture by the 45 ° line

Out[129]=



This corresponds to a partition of 5 = 4 + 1. So for each parton of n into parts of size at most r, we can do this flipping and get a partition of n into no more than r parts. We can also reversion this transformation. In other words, $a_n = b_n$ so they must have the same GF.

$$\prod_{m=1}^{r}\frac{1}{1-x^m}$$