Assignment 3 -- Solution

9.9.9 (c)

The solution for

$(-3+A)^3 f == (-27+27A-9A^2+A^3) f == 1+3n$

$\frac{1}{16}(-11-6n)+3^n c_1+3^n n c_2+3^n n^2 c_3$

The solution for the homogenous version

$-27+27A-9A^2+A^3$

is

$3^n c_1+3^n n c_2+3^n n^2 c_3$

We only need to find a particular solution for

$(A-3)^3 f == (A^3-9A^2+27A-27) f == 1+3n$

If we try $d_1 + d_2 n$, we get

$-8d_1+12d_2-8 n d_2 == 1+3n$

In other words,

$-8d_1+12d_2 == 1$

$-8d_2 == 3$

Solving this, we get

$\{\{d_1 \to -\frac{11}{16}, d_2 \to -\frac{3}{8}\}\}$

So one particular solution is

$\frac{1}{16}(-11-6n)$

And the general solution for the original problem is

$\frac{1}{16}(-11-6n)+3^n c_1+3^n n c_2+3^n n^2 c_3$

9.9.9 (g)

The general solution for the homogenous version is simply

$(-1)^n c_1+3^n c_2+3^n n c_3$

To get a particular solution, we can guess $(d_1 + d_2\, n + d_3\, n^2)\, 3^n$. This gives a particular solution

$$\frac{1}{32} \times 3^{-2+n} \left(9 - 20\, n + 8\, n^2\right)$$

Sum up the two expressions about and we get the

$$\frac{3^n}{32} - \frac{5}{8} \times 3^{-2+n}\, n + \frac{1}{4} \times 3^{-2+n}\, n^2 + (-1)^n\, c_1 + 3^n\, c_2 + 3^n\, n\, c_3$$

Note that this can be simplified to

$$\frac{1}{36} \times 3^n\, n^2 + (-1)^n\, c_1 + 3^n\, c_2 + 3^n\, n\, c_3 \qquad (1)$$

9.9.12

$g(n)$ satisfies the following recurrence

$$g[n] == g[n-1] + 6\, g[n-2] + 3^{n-1}$$

In other words

$$g[n] - g[n-1] - 6\, g[n-2] == 3^{n-1}$$

In terms of Advancement Operator,

$$\left(A^2 - A - 6\right)\, g = 3^{n+1}$$

The general solution for this is

$$-\frac{3^{1+n}}{25} + \frac{3^n\, n}{5} + (-2)^n\, c_1 + 3^n\, c_2$$

We are given $g[0] == 1$ and we know by counting that $g[1] == 2$. So we get

$$\left\{\left\{c_1 \to \frac{8}{25},\; c_2 \to \frac{4}{5}\right\}\right\}$$

and

$$g[n] == \frac{1}{25}\, (-1)^n\, 2^{3+n} + \frac{17 \times 3^n}{25} + \frac{3^n\, n}{5}$$

9.9.18

Fibonacci sequence satisfies that $f[0] == 0$, $f[1] == 1$, and for $n \geq 0$

$$f[n+2] == f[n+1] + f[n]$$

Let $F[x] == \sum_{n=0}^{\infty} f[n]\, x^n$. Then if we sum up

$$f[n+2]\, x == f[n+1]\, x + f[n]\, x$$

for $n \geq 0$, we get

$$\sum_{n=0}^{\infty} f[n+2]\ x == \sum_{n=0}^{\infty} f[n+1]\ x + \sum_{n=0}^{\infty} f[n]\ x$$

which is equivalent to

$$\frac{1}{x^2}\,(F[x] - f[0] - f[1]\ x) == \frac{1}{x}\,(F[x] - f[0]) + F[x]$$

Solve this we get

$$F[x] == -\frac{x}{-1 + x + x^2}$$

If we turn this into Partial Fraction, we get

$$F[x] == -\frac{x}{-1 + x + x^2} == -\frac{x}{\left(\frac{1}{2}\left(1 - \sqrt{5}\right) + x\right)\left(\frac{1}{2}\left(1 + \sqrt{5}\right) + x\right)} ==$$

$$\frac{-1 + \sqrt{5}}{\sqrt{5}\left(-1 + \sqrt{5} - 2\,x\right)} + \frac{-1 - \sqrt{5}}{\sqrt{5}\left(1 + \sqrt{5} + 2\,x\right)}$$

which is

$$\frac{1}{\sqrt{5}}\left(\frac{1}{\left(1 - \frac{2}{-1+\sqrt{5}}\,x\right)} - \frac{1}{\left(1 + \frac{2}{1+\sqrt{5}}\,x\right)}\right)$$

So by Newton’s binomial expansion

$$f(n) = \frac{\left(-\frac{2}{\sqrt{5}-1}\right)^n \binom{-1}{n} - \left(\frac{2}{1+\sqrt{5}}\right)^n \binom{-1}{n}}{\sqrt{5}} = \frac{\left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^n - \left(\frac{1}{2}\left(1-\sqrt{5}\right)\right)^n}{\sqrt{5}}$$

For $n == 0 \ldots, 5$, this is

{0, 1, 1, 2, 3, 5}

So obviously we get the correct answer.

10.8.3

(a)

The probability of getting no “5” in three spins is $(5/8)^3$. So the probability of getting at least one “5” is

$$1 - \left(\frac{5}{8}\right)^3$$

(b)

The same argument as in (a). The answer is

$$1 - \left(\frac{7}{8}\right)^3$$

(c)

Let A be the event of getting a “2”. Let B be the event of getting a “5” or a “2”. Then

$$P(A \mid B) == \frac{P(A \cap B)}{P(B)} = \frac{2/8}{5/8} = \frac{2}{5}$$

(d)

The expected value is

$$\frac{1}{8} 1 + \frac{2}{8} 2 + \frac{1}{8} 3 + \frac{1}{8} 4 + \frac{3}{8} 5 == \frac{27}{8} = 3.375$$

This is more than 3. So you should play the game by paying 3 dollars.

10.8.4

You can use the Theorem 7.12 which says the probability to get a derangement if you choose a uniform permutation of [n] is about $\frac{1}{e} \neq \frac{1}{2}$ for large n.

So the expect reward for the game is close to

$$\frac{5}{2} \frac{1}{e} - 1 \left(1 - \frac{1}{e}\right) == 0.287578$$

So this is not a fair game.

Let’s say we are paid x dollars to play this game. Then the expected reward for this game is very close to

$$x \frac{1}{e} - \left(1 - \frac{1}{e}\right)$$

Taking

$$x == -1 + e == 1.71828$$

makes the expected reward very close to 0, i.e., making it (very close to) fair.