1 – Introduction to Combinatorics

Combinatorics 1M020

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Problems studied in combinatorics

Combinatorics consists of many topics–

- Discrete Structures – graphs, strings, patterns, ...
- Enumeration – permutations, combinations, generating functions, recurrence relations, ...
- Algorithms and Optimization – sorting, shortest path, graph coloring, ...

We will only be able cover a tiny part of them 😒.

Today we will see many motivational problems! No proofs at all!

Enumeration

Example: 10 dollars, 3 children

Question

Amanda has three children: Dawn, Keesha and Seth. She has 10 one-dollar bills that she wants to give to the children. How many ways can she do it?

Example: 10 dollars, 3 children

If Amanda wants to give all $10 to her children.

Dawn	Keesha	Seth
💵 💵 💵 💵	💵 💵 💵	💵 💵 💵
💵 💵 💵 💵	💵 💵 💵 💵	💵 💵
💵 💵 💵 💵 💵 💵 💵 💵 💵 💵		
💵 💵 💵 💵 💵 💵 💵 💵	💵	💵

Table 1: Some possible ways to distribute $10 among 3 children

Example: 10 dollars, 3 children

If Amanda does not have to give all $10 to her children.

Dawn	Keesha	Seth
💵 💵 💵 💵	💵 💵 💵	💵 💵 💵
💵 💵 💵	💵 💵 💵	💵 💵 💵
💵 💵 💵	💵 💵	💵

Table 2: Some possible ways to distribute at most $10 among 3 children

Example: 10 dollars, 3 children

Note

Amanda do not distinguish two one-dollar bills. Only the amount each child gets matters for her.

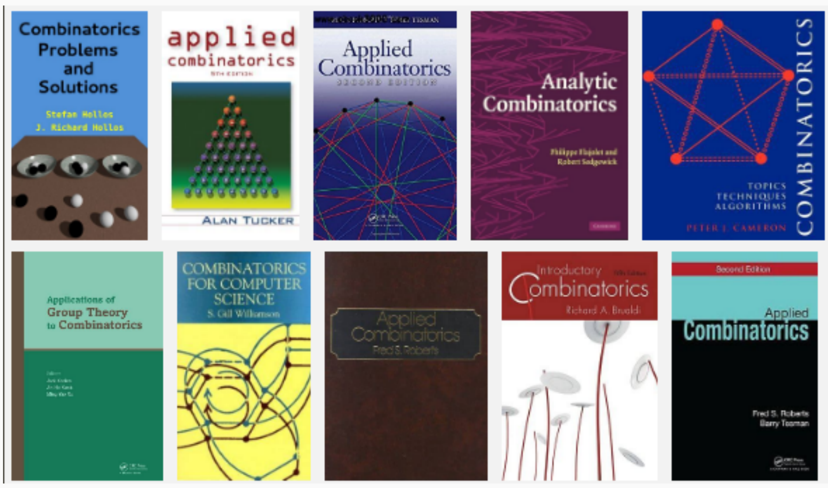
Quiz

What if Amanda wants to give two dollars to her three children. How many ways can she divide it among her children?

Example: 10 books, 3 children

Question

Amanda has has 10 different books 📕$_1$, 📕$_2$, … , 📕$_{10}$ that she wants to give to the children. How many ways can she do it?



Example: 10 books, 3 children

If Amanda wants to give all 10 books to her children.

Dawn	Keesha	Seth
📕$_1$📕$_2$📕$_3$📕$_4$	📕$_5$📕$_6$📕$_7$📕$_8$	📕$_9$📕$_{10}$
📕$_5$📕$_6$📕$_7$📕$_8$	📕$_1$📕$_2$📕$_3$📕$_4$	📕$_9$📕$_{10}$
📕$_1$📕$_2$📕$_3$📕$_4$📕$_5$ 📕$_6$📕$_7$📕$_8$📕$_9$📕$_{10}$		
📕$_1$📕$_2$📕$_3$📕$_4$ 📕$_5$📕$_6$📕$_7$📕$_8$	📕$_9$	📕$_{10}$

Table 3: Some possible ways to distribute 10 different books among 3 children

Example: 10 books, 3 children

If Amanda does not have to give all 10 books to her children.

Dawn	Keesha	Seth
📕$_1$📕$_2$📕$_4$	📕$_6$📕$_7$📕$_8$	📕$_9$📕$_{10}$
📕$_5$📕$_7$📕$_8$	📕$_2$📕$_3$📕$_4$	📕$_9$📕$_{10}$
📕$_1$📕$_2$📕$_3$📕$_4$📕$_5$ 📕$_6$📕$_7$📕$_8$📕$_9$		

Table 4: Some possible ways to distribute at most 10 different books among 3 children

Example: 10 books, 3 children

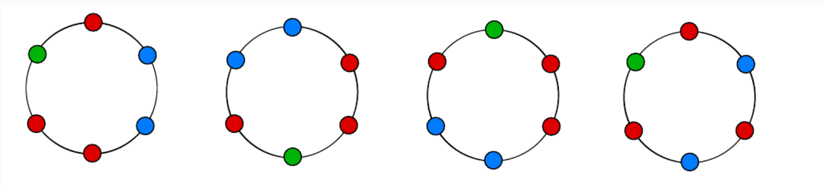
Note

Now which child gets which books matters, because the books are different.

Quiz

What if Amanda has only two different books?

Example: necklace



The first three necklaces are same! (Can you see it?)

The last one is different.

Given 3 red, 2 blue and 1 green beads, how many different necklaces can we make?

What if we do not need to use all of them?

Quiz

What about just 1 red, 1 blue and 1 green?

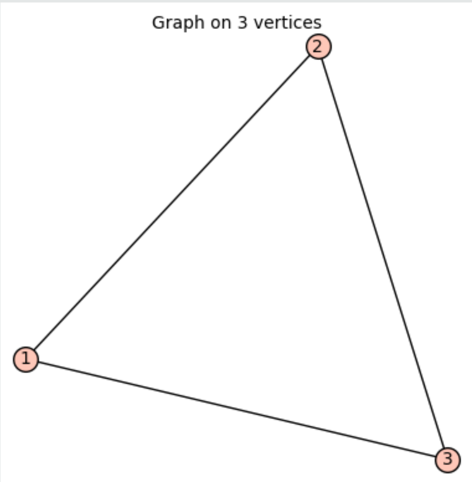
Graph Theory

What is a graph?

A graph G consists of a vertex set V and a collection E of 2-element subsets of V. Elements of E are called edges.

Example 1

$V = \{1, 2, 3\}$. $E = \{(1, 2), (1, 3), (2, 3)\}$.

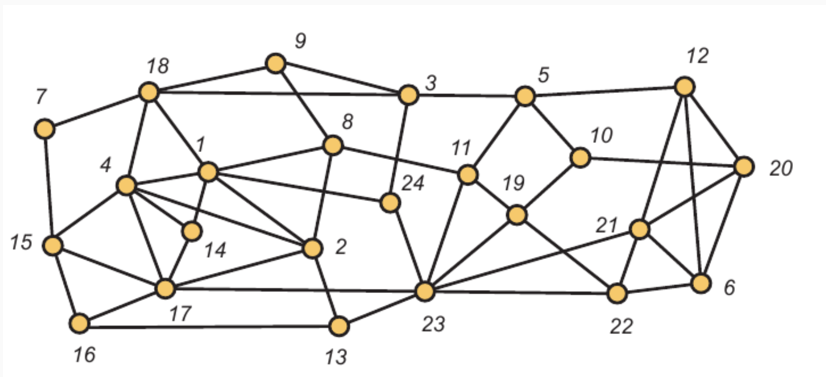


Example 2

A few words on Sagemath

The pictures are drawn with SageMath, a free open-source mathematics software system. Free online version at https://cocalc.com.
Not required in this course – only use for demonstrations.
Highly recommend for those who already known Python.

Questions in graph theory



Think the nodes as cities and the edges as roads, we can ask many question, e.g.:

- Can you travel through all cities without repetition?
- Can you travel through all cities without repetition and comes back to where you started? (Travel salesman problem)
- Can we build all the roads without crossing?

Number Theory

How do we add fractional numbers?

Quiz

$$\frac{2}{15} + \frac{7}{12} = ?$$

Answer

$$\frac{2}{15} + \frac{7}{12} = \frac{8}{60} + \frac{35}{60} = \frac{43}{60}.$$

We can do the addition by finding the least common multiple of 12 and 15, which is 60.

How hard is it to find the least common multiple of two integers?

This is easy if we can easily factor integers!

If we know

$$351785000 = 2^3 \times 5^4 \times 7 \times 19 \times 23^2$$

and

$$316752027900 = 2^2 \times 3 \times 5^2 \times 7^3 \times 11 \times 23^4.$$

Then their least common multiple is just

$$300914426505000 = 2^3 \times 3 \times 5^4 \times 7^3 \times 11 \times 19 \times 23^4.$$

How easy is it factor integers?

Quiz

Factor the integer

$$c = 556849011707703570824428317333504052171636923\backslash$$
$$558995115096520431388982368170755475721537 99$$

Try this in SageMath or WolframAlpha.

Already very hard for 88 digits!

How easy is it factor integers?

The answer

$$
\begin{aligned}
a &= 2425967623052370772757633156976982469681 \\
b &= 22953686867719691230002707821868552601124472329079 \\
c &= a \times b.
\end{aligned}
$$

Easy to verify.

Very difficult to find.

Internet security depends on integer factorization is hard!

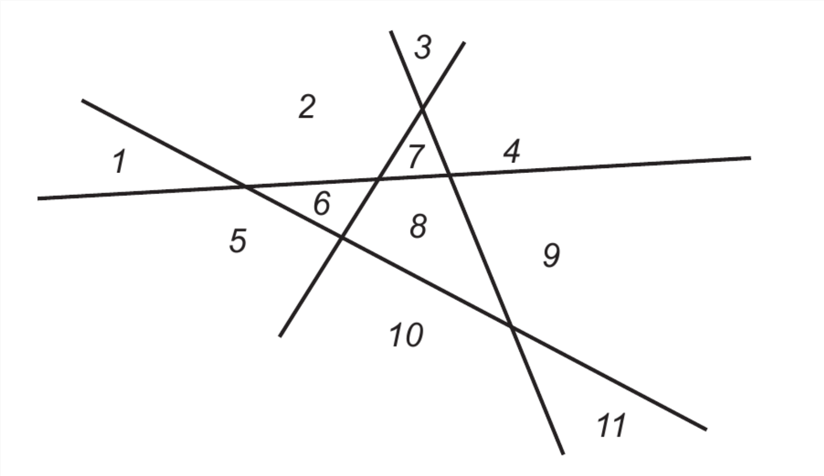
Geometry

Lines and areas

Each pair of the 4 lines intersects.

No point in the plane belongs to more than two lines.

These 4 lines determine 11 regions.



What about 8947 lines? How many regions will they determine?

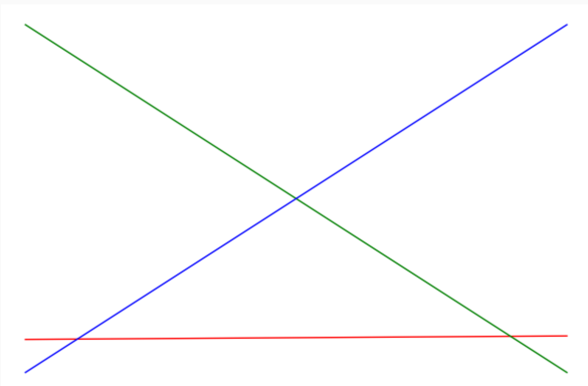
Lines and areas

Each pair of the n lines intersects.

No point in the plane belongs to more than two lines.

Quiz

How many regions does 2 lines determine? How many regions does 3 lines determine? How many regions does 5 lines determine? Do you see a pattern here?



Collatz sequence

Start with a positive integer $n > 1$. If n is odd, next number is $3n + 1$. If n is even, next number is $n/2$.

Example

If we start with 28,

$$28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.$$

If we start with 19,

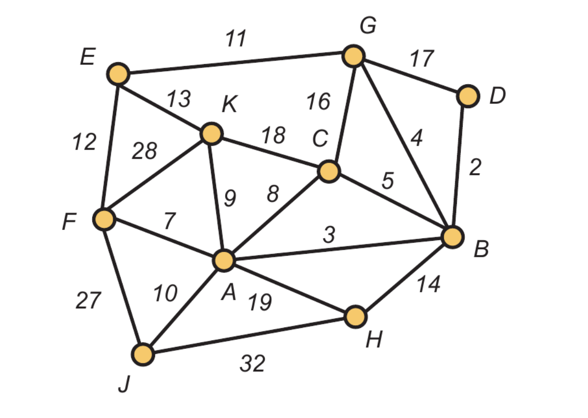
$$19, 58, 29, 88, 44, 22, \dots, 1$$

Problem

Is there n such that the sequence do not terminate? (No such $n < 87 \times 2^{60}$).

Optimization

Fining the shortest path



Assume the vertices are cities and the edges are roads. Assume the number on the edges are distance between cities.

Problem

If a postman want to visit all from starting from A, visit all cities, and come back to A. What is the shortest route that he can take?

Amazon/UPS/DHL cares about this problem a lot!

Sudoku

Sudoku problems

		7				8	2	
	9				1			
	4		9	7				
					5	4		6
		3				7		
5		6	7					
				8	4		5	
			6				1	
	2	4				6		

	8	1	3		2	6		
6		9	5		1		2	
2	3							
5		2		3		7	8	9
4	6	3		8		2		1
							6	2
	2		7		9	5		3
		6	8		3	9	4	

The game – filling up the 9x9 square with $1, \dots, 9$ such that each number appears in each row, each column, and each of the nine 3x3 sub-squares exactly once.

Sudoku in movies

Sometimes used in movies to show a person is smart!

Figure 1: Eye in the Sky (2007 film)

Sudoku problems

Solve a particular Sudoku is interesting.

But more interesting questions are

- How can we generate good Sudoku problems?
- How to solve them by computer?
- Given a Sudoku problem, how many ways can we fill it?
- ...

Quiz (Optional)

Try to solve the two Sudoku problems with SageMath.

Appendix

Attention Paraskevidekatriaphobes

Paraskevidekatriaphobia – noun [uncountable] fear of Friday the thirteenth.

Puzzle

Is the 13th of the month more likely to be a Friday than any other day of the week, or does it just seem that way? (*Mathematical Mind-Benders, by Peter Winkler*)

Remark

At the end of each lecture, if there is time, we will discuss a more recreational problem.

Attention Paraskevidekatriaphobes

Gregorian calendar has a 400-year cycle.

So we only need to count the number months whose 13-th day is a Friday among the 4800 months.

Better to do with computer!

Sometimes brute force counting do solve the problem.

Self-study guide

Read textbook Chap. 1, *An Introduction to Combinatorics*. Section 1.3 and 1.4 can be skipped.

Watch online video lectures at

http://pwp.gatech.edu/math3012openresources/lecture-videos/course-overview/

Think of the quiz about lines and areas for 5 lines.