10 – Probability Part (2)

Combinatorics 1M020

Xing Shi Cai

08-03-2019

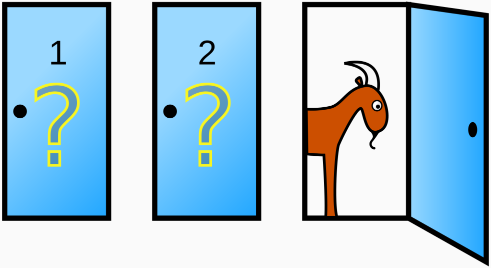
Department of Mathematics, Uppsala University, Sweden

The Monty Hall Problem

The Monty Hall Problem

Problem

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?



Two wrong answers

Answer 1: Among the remaining two doors, both are equally likely to have the car. So it does not matter if we switch or not.

Two wrong answers

Answer 1: Among the remaining two doors, both are equally likely to have the car. So it does not matter if we switch or not.

Answer 2: If, regardless of the host's action, the player's strategy is to never switch, she will obviously win the car $1/3$ of the time. Hence the probability that she wins if she does switch is $2/3$.

The probability space

The possible outcomes are

$$S = \{AGG2, AGG3, GAG2, GAG3, GGA2, GGA3\}$$

The probability space

The possible outcomes are

$$S = \{AGG2, AGG3, GAG2, GAG3, GGA2, GGA3\}$$

Let p_{ij} be the probability that the host opens door j given that the car is behind door j.

The probability space

The possible outcomes are

$$S = \{AGG2, AGG3, GAG2, GAG3, GGA2, GGA3\}$$

Let p_{ij} be the probability that the host opens door j given that the car is behind door j.

Since the host always opens either door 2 or door 3, $p_{i2} + p_{i3} = 1$.

The probability space

The possible outcomes are

$$S = \{AGG2, AGG3, GAG2, GAG3, GGA2, GGA3\}$$

Let p_{ij} be the probability that the host opens door j given that the car is behind door j.

Since the host always opens either door 2 or door 3, $p_{i2} + p_{i3} = 1$.

Also

$$P(AGG2) = P(AGG*) \times P(AGG2|AGG*) = \frac{1}{3}p_{12}$$

The probability space

The possible outcomes are

$$S = \{AGG2, AGG3, GAG2, GAG3, GGA2, GGA3\}$$

Let p_{ij} be the probability that the host opens door j given that the car is behind door j.

Since the host always opens either door 2 or door 3, $p_{i2} + p_{i3} = 1$.

Also

$$P(AGG2) = P(AGG*) \times P(AGG2|AGG*) = \frac{1}{3}p_{12}$$

and

$$P(AGG3) = P(AGG*) \times P(AGG3|AGG*) = \frac{1}{3}p_{13}$$

etc.

What probability are we looking for?

Let W_s be the event that we win by switching, i.e., $GAG3$.

What probability are we looking for?

Let W_s be the event that we win by switching, i.e., $GAG3$.

Let $D3$ be the event that door 3 is opened with a goat, i.e., $GAG3$ or $AGG3$.

What probability are we looking for?

Let W_s be the event that we win by switching, i.e., $GAG3$.

Let $D3$ be the event that door 3 is opened with a goat, i.e., $GAG3$ or $AGG3$.

Then

$$P(W_s|D3) = \frac{P(GAG3)}{P(GAG3, AGG3)} = \frac{\frac{1}{3}p_{23}}{\frac{1}{3}p_{23} + \frac{1}{3}p_{13}} = \frac{p_{23}}{p_{23} + p_{13}}$$

What probability are we looking for?

Let W_s be the event that we win by switching, i.e., $GAG3$.

Let $D3$ be the event that door 3 is opened with a goat, i.e., $GAG3$ or $AGG3$.

Then

$$P(W_s|D3) = \frac{P(GAG3)}{P(GAG3, AGG3)} = \frac{\frac{1}{3}p_{23}}{\frac{1}{3}p_{23} + \frac{1}{3}p_{13}} = \frac{p_{23}}{p_{23} + p_{13}}$$

Choosing

- $p_{23} = 0$ gives 0. (This is the case that if the car is behind door 2, then the host always opens the door 2 to end the game.)

What probability are we looking for?

Let W_s be the event that we win by switching, i.e., $GAG3$.

Let $D3$ be the event that door 3 is opened with a goat, i.e., $GAG3$ or $AGG3$.

Then

$$P(W_s|D3) = \frac{P(GAG3)}{P(GAG3, AGG3)} = \frac{\frac{1}{3}p_{23}}{\frac{1}{3}p_{23} + \frac{1}{3}p_{13}} = \frac{p_{23}}{p_{23} + p_{13}}$$

Choosing

- $p_{23} = 0$ gives 0. (This is the case that if the car is behind door 2, then the host always opens the door 2 to end the game.)
- $p_{23} = p_{13}$ gives $1/2$.

What probability are we looking for?

Let W_s be the event that we win by switching, i.e., $GAG3$.

Let $D3$ be the event that door 3 is opened with a goat, i.e., $GAG3$ or $AGG3$.

Then

$$P(W_s|D3) = \frac{P(GAG3)}{P(GAG3, AGG3)} = \frac{\frac{1}{3}p_{23}}{\frac{1}{3}p_{23} + \frac{1}{3}p_{13}} = \frac{p_{23}}{p_{23} + p_{13}}$$

Choosing

- $p_{23} = 0$ gives 0. (This is the case that if the car is behind door 2, then the host always opens the door 2 to end the game.)
- $p_{23} = p_{13}$ gives $1/2$.
- $p_{13} = 0$ gives 1.

What probability are we looking for?

Let W_s be the event that we win by switching, i.e., $GAG3$.

Let $D3$ be the event that door 3 is opened with a goat, i.e., $GAG3$ or $AGG3$.

Then

$$P(W_s|D3) = \frac{P(GAG3)}{P(GAG3, AGG3)} = \frac{\frac{1}{3}p_{23}}{\frac{1}{3}p_{23} + \frac{1}{3}p_{13}} = \frac{p_{23}}{p_{23} + p_{13}}$$

Choosing

- $p_{23} = 0$ gives 0. (This is the case that if the car is behind door 2, then the host always opens the door 2 to end the game.)
- $p_{23} = p_{13}$ gives $1/2$.
- $p_{13} = 0$ gives 1.

Impossible to decide if to switch without knowing p_{23} and p_{13}.

With a bit more information

What if the host is not allowed to open the door with the car?

With a bit more information

What if the host is not allowed to open the door with the car? In other words

$$p_{22} = p_{33} = 0, \quad p_{23} = p_{32} = 1.$$

With a bit more information

What if the host is not allowed to open the door with the car? In other words

$$p_{22} = p_{33} = 0, \quad p_{23} = p_{32} = 1.$$

Then

$$P(W_s|D3) = \frac{p_{23}}{p_{23} + p_{13}} = \frac{1}{1 + p_{13}} \geq \frac{1}{2}$$

since $p_{13} \leq 1$.

With a bit more information

What if the host is not allowed to open the door with the car? In other words

$$p_{22} = p_{33} = 0, \quad p_{23} = p_{32} = 1.$$

Then

$$P(W_s|D3) = \frac{p_{23}}{p_{23} + p_{13}} = \frac{1}{1 + p_{13}} \geq \frac{1}{2}$$

since $p_{13} \leq 1$.

So we should always switch!

Bernoulli Trials

Bernoulli Trails

An experiment has two out comes: success and failure.

Bernoulli Trails

An experiment has two out comes: success and failure.

The probability of success is p and the probability of failure is $1 - p$.

Bernoulli Trails

An experiment has two out comes: success and failure.

The probability of success is p and the probability of failure is $1-p$.

If we repeat the experiment, the outcome of each experiment is independent.

Bernoulli Trails

An experiment has two out comes: success and failure.

The probability of success is p and the probability of failure is $1-p$.

If we repeat the experiment, the outcome of each experiment is independent.

Quiz What is the probability of n such experiments with i success and $n-i$ failure?

Bernoulli Trails

An experiment has two out comes: success and failure.

The probability of success is p and the probability of failure is $1-p$.

If we repeat the experiment, the outcome of each experiment is independent.

Quiz What is the probability of n such experiments with i success and $n-i$ failure?

$$P(i \text{ success in } n \text{ trials}) = x_i = \binom{n}{i} p^i (1-p)^{n-i}.$$

Bernoulli Trails

An experiment has two out comes: success and failure.

The probability of success is p and the probability of failure is $1-p$.

If we repeat the experiment, the outcome of each experiment is independent.

Quiz What is the probability of n such experiments with i success and $n-i$ failure?

$$P(i \text{ success in } n \text{ trials}) = x_i = \binom{n}{i} p^i (1-p)^{n-i}.$$

Quiz Why is $\sum_{i \geq 0} x_i = 1$?

Example – A fair coin

If we toss a fair coin, the probability of getting a head or a tail are both $1/2$.

Example – A fair coin

If we toss a fair coin, the probability of getting a head or a tail are both $1/2$.

Quiz If we toss a fair coin 100 times, what is the probability that we get 50 head and 50 tails?

Example – A fair coin

If we toss a fair coin, the probability of getting a head or a tail are both $1/2$.

Quiz If we toss a fair coin 100 times, what is the probability that we get 50 head and 50 tails?

$$\binom{100}{50}\left(\frac{1}{2}\right)^{50}\left(\frac{1}{2}\right)^{50}$$

Example – A fair coin

If we toss a fair coin, the probability of getting a head or a tail are both $1/2$.

Quiz If we toss a fair coin 100 times, what is the probability that we get 50 head and 50 tails?

$$\binom{100}{50}\left(\frac{1}{2}\right)^{50}\left(\frac{1}{2}\right)^{50}$$

Is this > 0.1?

Example – A fair coin

This probability is actually quite small

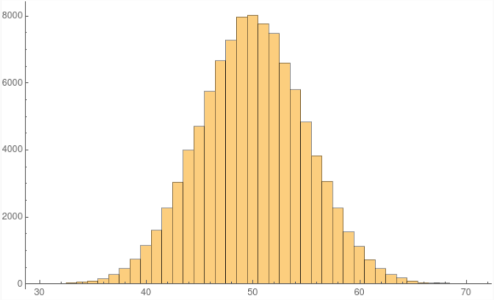
$$\binom{100}{50}\left(\frac{1}{2}\right)^{50}\left(\frac{1}{2}\right)^{50} \approx .079589$$

Example – A fair coin

This probability is actually quite small

$$\binom{100}{50}\left(\frac{1}{2}\right)^{50}\left(\frac{1}{2}\right)^{50} \approx .079589$$

But if we try to repeat this many times, the outcome is often close to 50. The picture is this experiment repeated 10^5 times.



Random variables

Random variables

Let (S, P) be a probability space. Let $X : S \to \mathbb{R}$. X is called a random variable.

Random variables

Let (S, P) be a probability space. Let $X : S \to \mathbb{R}$. X is called a random variable.

Example – we toss two fair dice. There are $6^2 = 36$ possible outcomes

$$S = \{(1,1), (1,2), \dots, (6,5), (6,6)\}.$$

Random variables

Let (S, P) be a probability space. Let $X : S \to \mathbb{R}$. X is called a random variable.

Example – we toss two fair dice. There are $6^2 = 36$ possible outcomes

$$S = \{(1,1), (1,2), \ldots, (6,5), (6,6)\}.$$

Because the dice are fair, each outcome is equally likely. So

$$P(s) = \frac{1}{36}, \qquad s \in S.$$

Random variables

Let (S, P) be a probability space. Let $X : S \to \mathbb{R}$. X is called a random variable.

Example – we toss two fair dice. There are $6^2 = 36$ possible outcomes

$$S = \{(1,1),(1,2),\dots,(6,5),(6,6)\}.$$

Because the dice are fair, each outcome is equally likely. So

$$P(s) = \frac{1}{36}, \qquad s \in S.$$

We can define a random variable

$$X((s_1, s_2)) = s_1 + s_2, \qquad (s_1, s_2) \in S.$$

Random variables

Let (S, P) be a probability space. Let $X : S \to \mathbb{R}$. X is called a random variable.

Example – we toss two fair dice. There are $6^2 = 36$ possible outcomes

$$S = \{(1,1), (1,2), \dots, (6,5), (6,6)\}.$$

Because the dice are fair, each outcome is equally likely. So

$$P(s) = \frac{1}{36}, \qquad s \in S.$$

We can define a random variable

$$X((s_1, s_2)) = s_1 + s_2, \qquad (s_1, s_2) \in S.$$

For example

$$X((3,6)) = 3 + 6 = 9.$$

Expectation of random variables

Expectation

The expectation or mean of X is defined by

$$E(X) = \sum_{x \in S} X(x)P(x)$$

Expectation

The expectation or mean of X is defined by

$$E(X) = \sum_{x \in S} X(x)P(x)$$

Think this as "on average what X should be".

Expectation

The expectation or mean of X is defined by

$$E(X) = \sum_{x \in S} X(x)P(x)$$

Think this as "on average what X should be".

In the example of fair dice

$$\begin{aligned} E(X) &= \sum_{(s_1, s_2) \in S} P((s_1, s_2))X((s_1, s_2)), \\ &= \frac{1}{36}(1+1) + \frac{1}{36}(1+2) + \cdots + \frac{1}{36}(5+6) + \frac{1}{36}(5+6) = ? \end{aligned}$$

Expectation

The expectation or mean of X is defined by

$$E(X) = \sum_{x \in S} X(x)P(x)$$

Expectation

The expectation or mean of X is defined by

$$E(X) = \sum_{x \in S} X(x)P(x)$$

or equivalently

$$E(X) = \sum_{y} y \times P(X = y)$$

where the sum is over all possible values of X.

Expectation

The expectation or mean of X is defined by

$$E(X) = \sum_{x \in S} X(x)P(x)$$

or equivalently

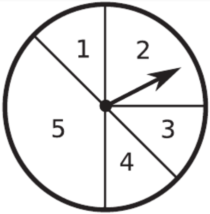
$$E(X) = \sum_{y} y \times P\left(X = y\right)$$

where the sum is over all possible values of X.

In the example of fair dice

$$\begin{aligned} E(X) &= \sum_{y=2}^{12} P(X = y)y \\ &= P(X = 2)2 + \cdots + P(X = 12)12 \end{aligned}$$

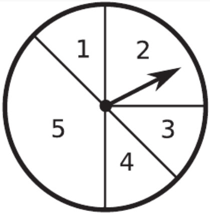
Example – spinner



If we let $X_1(i) = i$ where i is the landing region, then

$$E(X_1) = 1\frac{1}{8} + 2\frac{1}{4} + 3\frac{1}{8} + 4\frac{1}{8} + 5\frac{3}{8}.$$

Example – spinner



If we let $X_1(i) = i$ where i is the landing region, then

$$E(X_1) = 1\frac{1}{8} + 2\frac{1}{4} + 3\frac{1}{8} + 4\frac{1}{8} + 5\frac{3}{8}.$$

If we let $X(i)_2 = i^2$ where i is the landing region, then

$$E(X_2) = 1^2\frac{1}{8} + 2^2\frac{1}{4} + 3^2\frac{1}{8} + 4^2\frac{1}{8} + 5^2\frac{3}{8}.$$

Linearity of expectation

If $X_1, \ldots, X_n$ are random variables defined on the same probability space, then

$$E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n).$$

Linearity of expectation

If $X_1, \ldots, X_n$ are random variables defined on the same probability space, then

$$E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n).$$

In the example spinner

$$\begin{aligned} E(X_1) + E(X_2) = {} & 1\frac{1}{8} + 2\frac{1}{4} + 3\frac{1}{8} + 4\frac{1}{8} + 5\frac{3}{8} \\ & + 1^2\frac{1}{8} + 2^2\frac{1}{4} + 3^2\frac{1}{8} + 4^2\frac{1}{8} + 5^2\frac{3}{8} \end{aligned}$$

Linearity of expectation

If $X_1, \ldots, X_n$ are random variables defined on the same probability space, then

$$E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n).$$

In the example spinner

$$\begin{aligned} E(X_1) + E(X_2) &= 1\frac{1}{8} + 2\frac{1}{4} + 3\frac{1}{8} + 4\frac{1}{8} + 5\frac{3}{8} \\ &\quad + 1^2\frac{1}{8} + 2^2\frac{1}{4} + 3^2\frac{1}{8} + 4^2\frac{1}{8} + 5^2\frac{3}{8} \end{aligned}$$

And

$$\begin{aligned} E(X_1 + X_2) &= (1 + 1^2)\frac{1}{8} + (2 + 2^2)\frac{1}{4} + (3 + 3^2)\frac{1}{8} \\ &\quad + (4 + 4^2)\frac{1}{8} + (5 + 5^2)\frac{3}{8} \end{aligned}$$

Linearity of expectation

If $X_1, \dots, X_n$ are random variables defined on the same probability space, then

$$E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n).$$

Linearity of expectation

If $X_1, \dots, X_n$ are random variables defined on the same probability space, then

$$E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n).$$

In the example fair dice, let X_1 and X_2 be the outcome of the first and the second dice. Then

$$\begin{aligned} E(X_1) &= E(X_2) \\ &= \frac{1}{6}1 + \frac{1}{6}2 + \frac{1}{6}3 + \frac{1}{6}4 + \frac{1}{6}5 + \frac{1}{6}6 = 3.5 \end{aligned}$$

Linearity of expectation

If $X_1, \dots, X_n$ are random variables defined on the same probability space, then

$$E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n).$$

In the example fair dice, let X_1 and X_2 be the outcome of the first and the second dice. Then

$$\begin{aligned} E(X_1) &= E(X_2) \\ &= \frac{1}{6}1 + \frac{1}{6}2 + \frac{1}{6}3 + \frac{1}{6}4 + \frac{1}{6}5 + \frac{1}{6}6 = 3.5 \end{aligned}$$

And

$$E(X) = E(X_1 + X_2) = 2 \times 3.5 = 7$$

Bernoulli Trials

Consider n Bernoulli Trials with p success probability. Let X be the number of success.

Bernoulli Trials

Consider n Bernoulli Trials with p success probability. Let X be the number of success.

Then

$$E(X) = \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i} = np$$

Bernoulli Trials

Consider n Bernoulli Trials with p success probability. Let X be the number of success.

Then

$$E(X) = \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i} = np$$

Let $X_j = 0$ if the j-th trial fails and $X_j = 1$ otherwise.

Quiz What is $E(X_j)$?

Bernoulli Trials

Consider n Bernoulli Trials with p success probability. Let X be the number of success.

Then

$$E(X) = \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i} = np$$

Let $X_j = 0$ if the j-th trial fails and $X_j = 1$ otherwise.

Quiz What is $E(X_j)$?

$$E(X_j) = p \times 1 + (1-p) \times 0 = p$$

Bernoulli Trials

Consider n Bernoulli Trials with p success probability. Let X be the number of success.

Then

$$E(X) = \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i} = np$$

Let $X_j = 0$ if the j-th trial fails and $X_j = 1$ otherwise.

Quiz What is $E(X_j)$?

$$E(X_j) = p \times 1 + (1-p) \times 0 = p$$

Since $X = \sum_{j=1}^{n} X_j$,

$$E(X) = E\left(\sum_{j=1}^{n} X_i\right) = \sum_{j=1}^{n} E\left(X_j\right) = np.$$

Variance of random variables

Variance

For a random variable X, the variance of X is

$$\operatorname{var}(X) = E\left((X - E(X))^2\right) = E\left(X^2\right) - \left(E\left(X\right)\right)^2.$$

Variance

For a random variable X, the variance of X is

$$\mathrm{var}\left(X\right) = E\left(\left(X - E(X)\right)^2\right) = E\left(X^2\right) - \left(E\left(X\right)\right)^2.$$

The standard deviation of X is $\sigma_X = \sqrt{\mathrm{var}(X)}$.

Variance

For a random variable X, the variance of X is

$$\operatorname{var}(X) = E\left((X - E(X))^2\right) = E\left(X^2\right) - \left(E\left(X\right)\right)^2.$$

The standard deviation of X is $\sigma_X = \sqrt{\operatorname{var}(X)}$.

Bernoulli Trial

Let $X = 0$ if the trial fails and $X = 1$ otherwise. Then

$$\operatorname{var}(X) = E(X^2) - (E(X))^2 = p - p^2 = p(1 - p).$$

Think variance as "how likely is X far away from its expectation"

Variance – fair die

Let X be the outcome of throwing a fair die.

Variance – fair die

Let X be the outcome of throwing a fair die. Then

$$E(X) = \frac{1}{6}1 + \frac{1}{6}2 + \frac{1}{6}3 + \frac{1}{6}4 + \frac{1}{6}5 + \frac{1}{6}6 = \frac{7}{2}$$

Variance – fair die

Let X be the outcome of throwing a fair die. Then

$$E(X) = \frac{1}{6}1 + \frac{1}{6}2 + \frac{1}{6}3 + \frac{1}{6}4 + \frac{1}{6}5 + \frac{1}{6}6 = \frac{7}{2}$$

and

$$E(X^2) = \frac{1}{6}1 + \frac{1}{6}2^2 + \frac{1}{6}3^2 + \frac{1}{6}4^2 + \frac{1}{6}5^2 + \frac{1}{6}6^2 = \frac{91}{6}$$

Variance – fair die

Let X be the outcome of throwing a fair die. Then

$$E(X) = \frac{1}{6}1 + \frac{1}{6}2 + \frac{1}{6}3 + \frac{1}{6}4 + \frac{1}{6}5 + \frac{1}{6}6 = \frac{7}{2}$$

and

$$E(X^2) = \frac{1}{6}1 + \frac{1}{6}2^2 + \frac{1}{6}3^2 + \frac{1}{6}4^2 + \frac{1}{6}5^2 + \frac{1}{6}6^2 = \frac{91}{6}$$

So the variance of X is

$$\mathrm{var}\,(X) = E(X^2) - (E(X))^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

Independent random variables

Independent random variables

The random variables $X_1, \ldots, X_n$ are independent if all $1 \leq i < j \leq n$ and each pair a, b, the events $X_i = a$ and $X_j = b$ are independent.

Independent random variables

The random variables $X_1, \ldots, X_n$ are independent if all $1 \leq i < j \leq n$ and each pair a, b, the events $X_i = a$ and $X_j = b$ are independent.

For example, X_1, X_2 are independent if

$$P(X_1 = a \cap X_2 = b) = P(X_1 = a)P(X_2 = b),$$

for all a, b.

Independent random variables – Example

Let X_1 and X_2 be the outcome of throwing a die twice. Then X_1 and X_2 are independent.

Independent random variables – Example

Let X_1 and X_2 be the outcome of throwing a die twice. Then X_1 and X_2 are independent. For example

$$P(X_1 = 3, X_2 = 6) = \frac{1}{36}$$

Independent random variables – Example

Let X_1 and X_2 be the outcome of throwing a die twice. Then X_1 and X_2 are independent. For example

$$P(X_1 = 3, X_2 = 6) = \frac{1}{36}$$

and

$$P(X_1 = 3)P(X_2 = 6) = \frac{1}{6}\frac{1}{6} = \frac{1}{36}$$

Independent random variables – Example

Let X_1 and X_2 be the outcome of throwing a die twice. Then X_1 and X_2 are independent. For example

$$P(X_1 = 3, X_2 = 6) = \frac{1}{36}$$

and

$$P(X_1 = 3)P(X_2 = 6) = \frac{1}{6}\frac{1}{6} = \frac{1}{36}$$

Easy to see

$$P(X_1 = a, X_2 = b) = P(X_1 = a)P(X_2 = b) = \frac{1}{36}.$$

for $0 \leq a \leq 6$ and $0 \leq b \leq 6$.

Dependent random variables – Example

Let X_1 be the outcome of throwing a die. Let $X_2 = X_1$. Then X_1 and X_2 are not independent.

Dependent random variables – Example

Let X_1 be the outcome of throwing a die. Let $X_2 = X_1$. Then X_1 and X_2 are not independent.

For example

$$P(X_1 = 3, X_2 = 6) = 0$$

Dependent random variables – Example

Let X_1 be the outcome of throwing a die. Let $X_2 = X_1$. Then X_1 and X_2 are not independent.

For example

$$P(X_1 = 3, X_2 = 6) = 0$$

and

$$P(X_1 = 3)P(X_2 = 6) = \frac{1}{6}\frac{1}{6} = \frac{1}{36}$$

Product of independent variables

If $X_1, \dots, X_n$ are independent, then

$$E(X_1 X_2 \dots X_n) = E(X_1)E(X_2) \dots E(X_n)$$

Product of independent variables

If $X_1, \ldots, X_n$ are independent, then

$$E(X_1 X_2 \ldots X_n) = E(X_1)E(X_2) \ldots E(X_n)$$

For example, in the example of throwing a fair die twice,

$$\begin{aligned}
E(X_1 X_2) &= \sum_{x_1=1}^{6} \sum_{x_2=1}^{6} P\left(X_1 = x_1, X_2 = x_2\right) x_1 x_2 \\
&= \sum_{x_1=1}^{6} \sum_{x_2=1}^{6} P\left(X_1 = x_1\right) P\left(X_2 = x_2\right) x_1 x_2 \\
&= \left(\sum_{x_1=1}^{6} P\left(X_1 = x_1\right) x_1 \right) \left(\sum_{x_2=1}^{6} P\left(X_2 = x_2\right) x_2 \right) \\
&= E(X_1)E(X_2)
\end{aligned}$$

Sum of independent variables

If $X_1, \ldots, X_n$ are independent, then

$$\mathrm{var}(X_1 + X_2 + \cdots + X_n) = \mathrm{var}(X_1) + \mathrm{var}(X_2) + \cdots + \mathrm{var}(X_n)$$

Sum of independent variables

If $X_1, \dots, X_n$ are independent, then

$$\operatorname{var}(X_1 + X_2 + \cdots + X_n) = \operatorname{var}(X_1) + \operatorname{var}(X_2) + \cdots + \operatorname{var}(X_n)$$

For example

$$\begin{aligned}\operatorname{var}(X_1 + X_2) &= E((X_1 + X_2)^2) - (E(X_1 + X_2))^2 \\ &= \operatorname{var}(X_1) + \operatorname{var}(X_2) + E(2X_1X_2) - 2E(X_1)E(X_2) \\ &= \operatorname{var}(X_1) + \operatorname{var}(X_2).\end{aligned}$$

Not necessarily true if X_1 and X_2 are dependent.

Variance–Bernoulli Trials

Consider n Bernoulli Trials with p success probability. Let X be the number of success.

Variance–Bernoulli Trials

Consider n Bernoulli Trials with p success probability. Let X be the number of success.

Then

$$\mathrm{var}(X) = \sum_{i=0}^{n} (i - (np))^2 \binom{n}{i} p^i (1-p)^{n-i} = np(1-p)$$

Variance–Bernoulli Trials

Consider n Bernoulli Trials with p success probability. Let X be the number of success.

Then

$$\mathrm{var}(X) = \sum_{i=0}^{n} (i - (np))^2 \binom{n}{i} p^i (1-p)^{n-i} = np(1-p)$$

Let $X_j = 1$ if the j-th trial fails and $X_j = 1$ otherwise.

Variance–Bernoulli Trials

Consider n Bernoulli Trials with p success probability. Let X be the number of success.

Then

$$\mathrm{var}(X) = \sum_{i=0}^{n} (i - (np))^2 \binom{n}{i} p^i (1-p)^{n-i} = np(1-p)$$

Let $X_j = 1$ if the j-th trial fails and $X_j = 1$ otherwise.

Since $X = \sum_{i=1}^{n} X_j$, and $X_1, \ldots, X_n$ are independent,

$$\mathrm{var}(X) = \mathrm{var}\left(\sum_{j=1}^{n} X_i\right) = \sum_{j=1}^{n} \mathrm{var}\left(X_j\right) = np(1-p).$$

Appendix

Self-study guide (for people who missed the class)

- Watch online video lectures here.
- Read textbook chapter 10.3-10.5