2 – Strings and Binomial Coefficients (Part 1)

Combinatorics 1M020

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Strings

Set

A set is a collection of distinct objects. (Do not ask me what is an object. 😅)

In this course we only deal with finite sets.

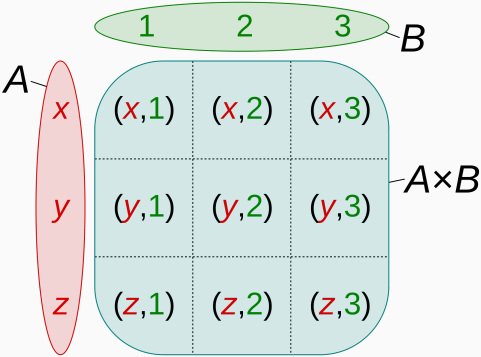
Examples of sets

- $\{a, b, c, d, e\}$.
- {💴, 💵, 💶, 💷}.
- {🍇, 🍈, 🍋, 🍉, 🍌}.

Most of the time, we will talk about $[n] = \{1, 2, \dots, n\}$.

Product of sets

The product of two sets $A \times B$ is the set of all pairs of (a, b) such that $a \in A$ and $b \in B$.



String

A sequence of length n like $(a_1, a_2, \dots, a_n)$ is called a string or word/vector/array/list.

The entries in a string are called characters/letters/coordinates.

The set of possible entries is called alphabet.

Examples

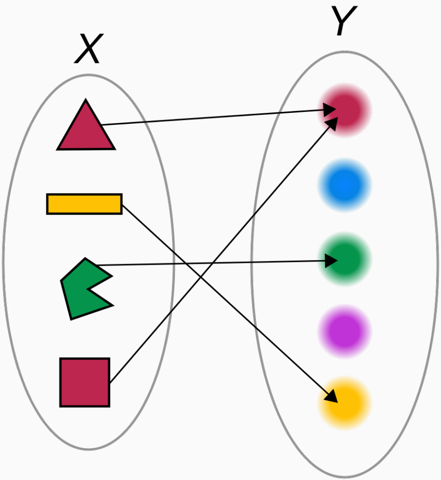
- 0100101000101100111 01 – a bit string
- 201002211001020 – a ternary string
- abcacbaccbbaaccbabaddbbadcabbd – a word from a four letter alphabet
- KSF 762 – an European vehicle license plate

More examples

- 🍇🍈🍇🍈🍋🍌🍉 – my breakfast of a week
- $(34, 53, 3, 43, 54, 64, 7)$ – a string from the set $[99]$.
- 蔡醒诗 – my name, a string from the set of all Chinese characters.
-  – a hand of playing cards is a string from the set of all 54 cards.

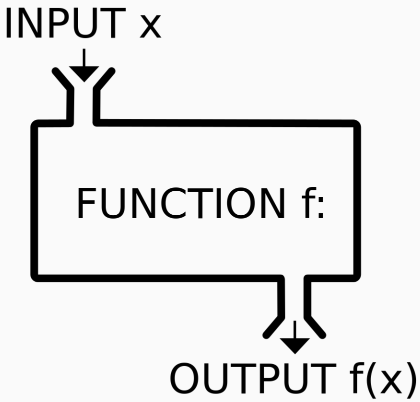
Function

A function is a mapping from one set to another.



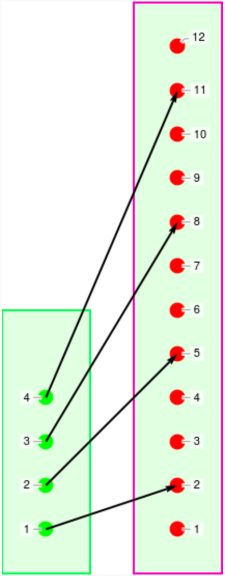
Function

A function can also be seen a rule to convert input to output.
(Just like a function in computer languages)



Notations of strings

- A string of length n on alphabet $\mathcal{A}$ is a function from $[n]$ to $\mathcal{A}$.
- The string $(2, 5, 8, 11)$ can be seen as a function $f : [4] \to [12]$ defined by $f(n) = 3n - 1$.
- Such a function (string) is often written as $(a_1, a_2, a_3, a_4, a_5, a_6)$ with $a_n = 3n - 1$.



String in computer languages (Python)

The string $(2, 5, 8, 11)$ can be represented as a list in Python

```
a = []
for i in range(1,4):
    a.append(3*i-1)
print a
```

Note that Python's lists (as well as arrays in C) start with index 0. So in the above example `a[0]==2`, `a[1]==5` and so on.

A basic principle of counting

If to finish a project has n steps, each step has m_i choices, then the total number of ways to do it is

$$m_1 \times m_2 \times m_3 \ldots m_n.$$

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Problem – How many different sandwiches?

Packing a sandwich has three steps

1. Choose bread from: 🥐, 🥖, 🥯.
2. Choose fillings from: tuna, ham, cheese, avocado.
3. Choose sauce from: ketchup, mayonnaise, soya sauce.

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Answer

$3 \cdot 4 \cdot 3 = 36$ different sandwiches.

Password

Problem

How many passwords satisfy

- The first letter is an upper-caser letter
- The second to the six characters must be a letter or a digit
- The seventh must be either @ or .
- The eighth through twelfth positions allow lower-case letters, *, %, and #.
- The thirteenth position must be a digit.

							#	#	#	#	#	
	D	D	D	D	D		%	%	%	%	%	
	L	L	L	L	L	.	*	*	*	*	*	
U	U	U	U	U	U	@	L	L	L	L	L	D
26	62	62	62	62	62	2	29	29	29	29	29	10

TABLE 2.4: STRING TEMPLATE

So the number of possible password is

$$26 \times 62^5 \times 2 \times 29^5 \times 10 = 9771287250890863360.$$

License plates in Sweden

Quiz

In Sweden, a vehicle plate has three letters, followed three digits. The letters I, Q, V, Å, Ä and Ö are not allowed. How many license plates are possible?

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Answer

Let X be the set of allowed letters. Then $|X| = 23$. Let Z be the set of digits. Then a plate number is a string from

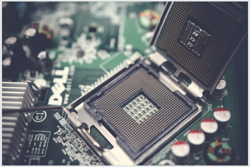
$$X \times X \times X \times Z \times Z \times Z.$$

So there are $23^3 \times 10^3 = 12167000$.

64 bit CPU

Quiz

A machine instruction for a 64-bit processor is a bit string of length 64. What is the number of such strings?



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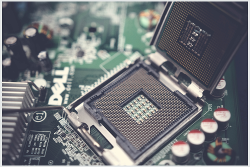

The number of bit strings of length n is 2^n.

The number of ternary strings of length n is 3^n.

The number of words of length n from an m letter alphabet is m^n .

Swedish personal number

Quiz – How many possible personal numbers for men?

- A personal identity number consists of 10 digits.
- The first 6 digits is the person's birthday, in YYMMDD.
- They are followed by three digits as a serial number.
- For the last digit, an odd number is assigned to males and an even number to females.

Think Why is 10^{10} the wrong answer?

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Think Why is 10^{10} the wrong answer?

Answer

Assuming each year has 365 days

$$100 \times 365 \times 1000 \times 5 = 182500000$$

Permutation

Example – letters from a bag

- Put the 26 letters of English alphabet in a bag.
- Take six letters out one by one, without replacement.
- This makes a string (word) of length six.

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Quiz

Could this word be yellow?

What is a permutation?

Definition

A permutation is a string without repetition.

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Example

12 7 8 6 4 9 11	Yes
X y a A D 7 B E 9	Yes
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The number of permutations of length n for an m-letter alphabet

$$P(m,n) = m(m-1)(m-2)\ldots(m-n+1)$$

Example of $P(m, n)$

Quiz

How many permutations of 23 letters taken from a 68-letter alphabet?

Example of $P(m, n)$

Quiz

How many permutations of 23 letters taken from a 68-letter alphabet?

Answer

$$P(68, 23) = 20732231223375515741894286164203929600000.$$

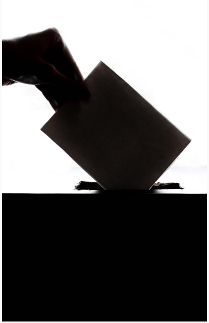
You do not have to compute the exact number it exams/assignment.

But if you are curious, try use SageMath.

Example – election

Quiz

A group of 40 students holds elections to identify a prime minister, a deputy prime minister, and a Minister for Finance. How many different outcomes are possible?



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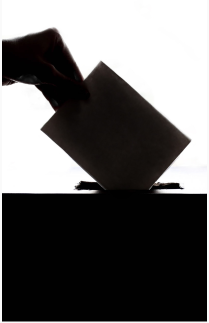

Answer

$$P(40, 3) = 40 \times 39 \times 38 = 59280.$$

License plates in Sweden - revisited

Problem

In Sweden, a vehicle plate has three letters, followed three digits. The letters I, Q, V, Å, Ä and Ö are not allowed.

In addition, the three letters cannot be the same.

How many license plates are possible?

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In addition, the three letters cannot be the same.

How many license plates are possible?

Answer

$$P(23,3) \times 10^3 = 10626000.$$

Combinations

Example – order food at a restaurant

Problem

A restaurant has 10 different dishes on its menu. We want to order 3 different dishes. How many different of combinations are possible?

In this problem, we do not care the order of the dishes.

Example – order food at a restaurant

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Answer

$$C(10,3) = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120.$$

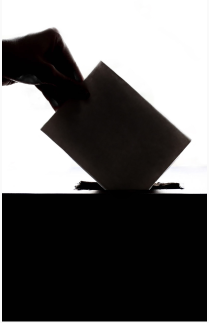
$C(m,n)$ is the number of combinations of n letter taken from an m alphabet.

Election – revisited

Quiz

A group of 40 students holds elections to form a class committee of three remembers.

How many different outcomes are possible?



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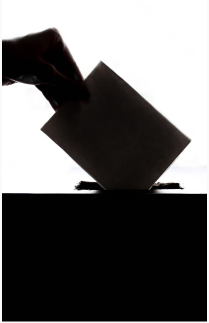

Answer

$$C(40,3) = \frac{40 \times 39 \times 38}{3 \times 2 \times 1} = 9880.$$

Binomial Coefficients

Another way to write

$$\binom{m}{n} = C(m, n).$$

$\binom{m}{n}$ reads as m choose n.

It is called a binomial coefficient. (We will see why soon!)

Factorial

We write $n! = n \times (n-1)(n-2)\ldots 1$.

This reads n factorial.

Quiz

Which one grows faster, $n!$ or 2^n?

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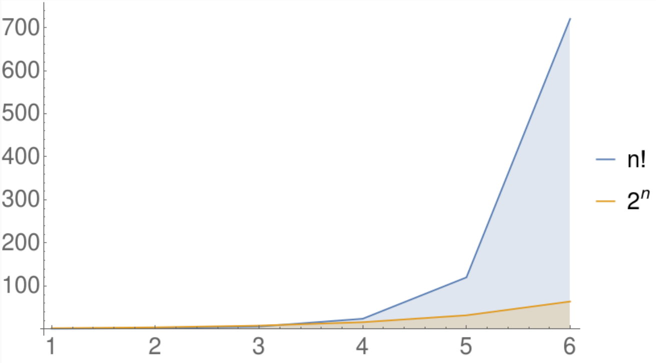

Compute binomial coefficient

Proposition 2.9

For $0 \leq n \leq m$

$$\binom{m}{n} = \frac{P(m,n)}{n!} = \frac{m(m-1)\cdots(m-n+1)}{n!} = \frac{m!}{n!(m-n)!}$$

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Quiz

Why is the $m!/(n!(m-n)!)$ an integer?

Combinatorial Proofs

Basic identities – complement

Quiz

What is $C(40, 39)$? Why compute it like $\frac{40!}{39!1!}$ is not the quickest way?

Basic identities – complement

Quiz

What is $C(40, 39)$? Why compute it like $\frac{40!}{39!1!}$ is not the quickest way?

Answer

This is simply $C(40, 1) = 40$. Choosing 39 out of 40 means there is 1 leftover.

Basic identities – complement

Quiz

What is $C(40, 39)$? Why compute it like $\frac{40!}{39!1!}$ is not the quickest way?

Answer

This is simply $C(40, 1) = 40$. Choosing 39 out of 40 means there is 1 leftover.

Proposition 2.10

For $0 \leq n \leq m$

$$C(m, n) = C(m, m - n)$$

Basic identities – recursion

Problem – Why is this true?

For $0 < n < m$

$$C(m, n) = C(m-1, n-1) + C(m-1, n)$$

Basic identities – recursion

Problem – Why is this true?

For $0 < n < m$

$$C(m,n) = C(m-1,n-1) + C(m-1,n)$$

Both sides count the number of m-element subsets of $[n]$.

The right-hand side first grouping them into those which contain the element n and then those which don't.

Basic identities – recursion

Problem – Why is this true?

For $0 < n < m$

$$C(m, n) = C(m-1, n-1) + C(m-1, n)$$

Both sides count the number of m-element subsets of $[n]$.

The right-hand side first grouping them into those which contain the element n and then those which don't.

Many identity can be checked by computer!

Pascal's triangle

A simple way to compute binomial coefficients

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

Pascal's triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

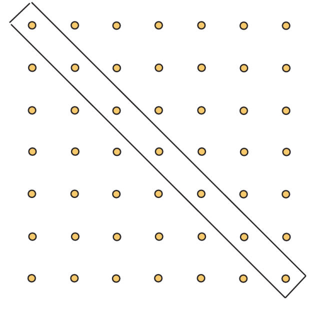
1 8 28 56 70 56 28 8 1

Table 3.2: Pascal's Triangle

See this nice picture on Wikipedia.

Combinatorial Proofs

- Combinatorial arguments are quite beautiful.
- Many statements can be proved by complicated methods.
- But often you can find very short proofs by counting.



Sum of the first n integers

Problem

How to prove

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

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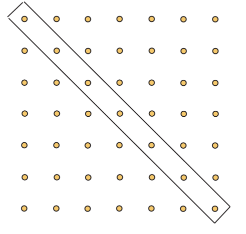
By this picture



$$1 + 2 + \cdots + n = \frac{(n+1)^2 - (n+1)}{2} = \frac{n(n+1)}{2}$$

Sum of the first n odd integers

Problem

How to prove

$$1 + 3 + \cdots + 2n - 1 = n^2$$

Problem

How to prove

$$1 + 3 + \cdots + 2n - 1 = n^2$$

By this picture

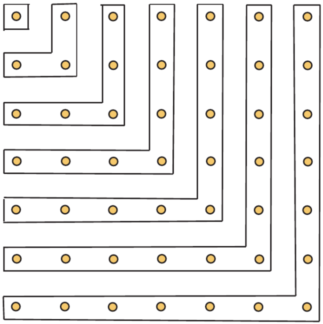


FIGURE 2.16: THE SUM OF THE FIRST n ODD INTEGERS

Combinatorial identities 1

Problem

$$C(2n, n) = C(n, 0)^2 + C(n, 1)^2 + ... C(n, n)^2$$

Combinatorial identities 1

Problem

$$C(2n,n) = C(n,0)^2 + C(n,1)^2 + \ldots C(n,n)^2$$

Both sides count the number of bit strings of length 2n with half the bits being 0's.

The right side first partitioning them according to the number of 1's occurring in the first n positions of the string.

Combinatorial identities 2

Problem

$$\binom{n}{k+1} = \binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n-1}{k}.$$

Combinatorial identities 2

Problem

$$\binom{n}{k+1} = \binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n-1}{k}.$$

Both sides count the number of bit strings of length n that contain $k+1$ 1's with the right hand side first partitioning them according to the last occurence of a 1.

Appendix

Snails on a circle

Nine 🐌 are on a circle of a 50 meter length.

At the start, each 🐌 decides randomly whether she would go, clockwise or counter-clockwise.

🐌 travel at speed 1 meter/minute.

When two 🐌 meet, they reverse direction.

After 100 minutes, we find the distances between the 🐌 are

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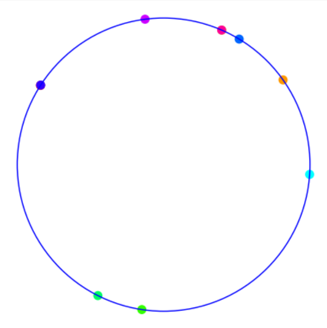
After 100 minutes, we find the distances between the 🐌 are

(Surprise!) exactly as before! Why!!?

Snails on a circle – Hint

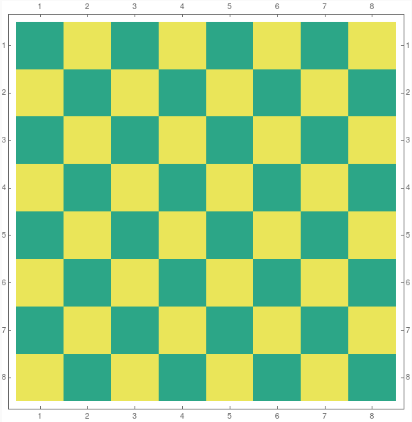


Assume that each snails is carrying a flag of a distinct color.

When two snails meet, they exchange the flags that they are carrying, then they reverse direction.

Where will the flags be after 100 minutes?

Tiling chessboard – Solution

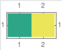


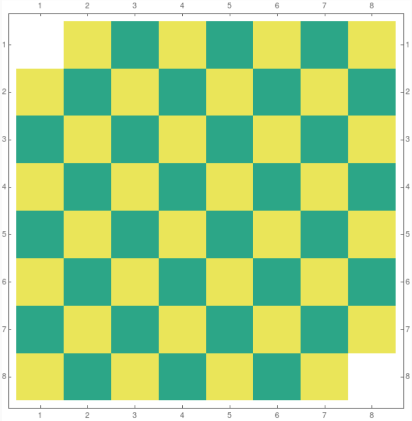
A chessboard has $8 \times 8 = 64$ squares.

32 of them are yellow. 32 of them are green.

Tiling chessboard – Solution

Quiz

This is a chessboard with two opposite corners removed. Can we completely cover it with 1x2 domino  ?



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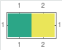
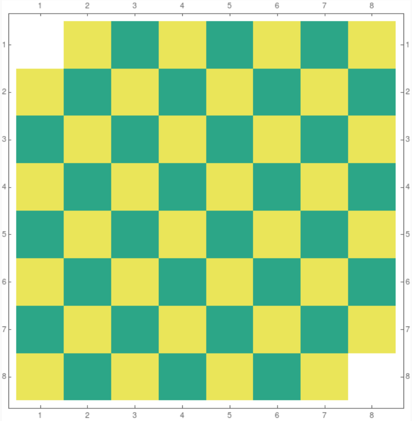
This is a chessboard with two opposite corners removed. Can we completely cover it with 1x2 domino  ?



Answer

The board has 32 of yellow and 30 green squares. Each domino covers 1 yellow 1 green.

Self-study guide – for yow who missed the class

- Read textbook 2.1-2.4.
- Watch online video lectures 1 to 5 here.
- Recommended exercises Have a quick look of
  - Textbook 2.9, 1–14
  - Online exercises here, 1-14