2 – Strings and Binomial Coefficients (Part 2)

Combinatorics 1M020

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Combinatorial proofs – continued

Combinatorial identities 3

Combinatorial arguments – proof by showing that both sides count the same thing.

Quiz

$$C(n,0) + C(n,1) + \cdots + C(n,n) = 2^n.$$

Combinatorial identities 3

Combinatorial arguments – proof by showing that both sides count the same thing.

Quiz

$$C(n,0) + C(n,1) + \cdots + C(n,n) = 2^n.$$

Quiz

$$C(n,0)2^0 + C(n,1)2^1 + \cdots + C(n,n)2^n = 3^n$$

Combinatorial identities 4

Problem

For $k > 0$,

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

Combinatorial identities 4

Problem

For $k > 0$,

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

This is equivalent to

$$\binom{k}{1}\binom{n}{k} = \binom{n}{1}\binom{n-1}{k-1}$$

Both sides count the number of strings of length n, containing one a, $k-1$ b's, and $n-k$ c's.

Combinatorial identities 5

Quiz

$$\sum_{i=1}^{n} \binom{n}{i} i = n2^{n-1}$$

Hint Both sides are the sum of the sizes of subsets of $[n]$.

Combinatorial identities 5

Quiz

$$\sum_{i=1}^{n} \binom{n}{i} i = n2^{n-1}$$

Hint Both sides are the sum of the sizes of subsets of $[n]$.

Answer

Observe that each element of $[n]$ is in 2^{n-1} subsets, and so contributes 2^{n-1} to the total sum. See here.

Combinatorial identities 6

Quiz – sum of squares

$$\sum_{k=1}^{n} k^2 = \binom{n+1}{2} + 2\binom{n+1}{3}$$

Hint Both sides count the number of ordered triples (i, j, k) with condition (what to put here?).

Combinatorial identities 6

Quiz – sum of squares

$$\sum_{k=1}^{n} k^2 = \binom{n+1}{2} + 2\binom{n+1}{3}$$

Hint Both sides count the number of ordered triples (i, j, k) with condition (what to put here?).

Answer

Both sides count the number of ordered triples (i, j, k) with condition $0 \leq i, j < k \leq n$. See here.

Combinatorial identities 7

Quiz – sum of squares

$$\sum_{k=1}^{n} k^3 = \binom{n+1}{2}^2$$

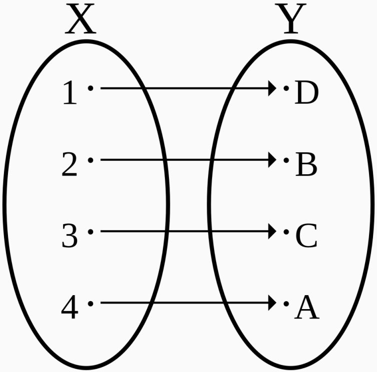
LHS counts the number of ordered 4-tuple (h, i, j, k) with condition $0 \leq h, i, j < k \leq n$.

RHS counts $(x_1, x_2), (x_3, x_4)$ with $0 \leq x_1 < x_2 \leq n$ and $0 \leq x_3 < x_4 \leq n$.

Need a bijection between both sides.

Bijections in combinatorics

If there is a bijection between two sets, then they must have the same number of elements.



Combinatorial identities 7 continued

A bijection between (h, i, j, k) and $(x_1, x_2), (x_3, x_4)$

- If $h < i$, then $(h, i), (j, k) \Leftrightarrow x_2 < x_4$.
- If $h > i$, then $(j, k), (i, h) \Leftrightarrow x_2 > x_4$.
- If $h = i$, then $(h, k), (j, k) \Leftrightarrow x_2 = x_4$.

See here.

A challenge for you

Problem

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{3n+k}{2n} = \binom{3n}{n}$$

Found on the T-shirt of Doron Zeilberger

I only know a computer proof.

Can you find a human proof? A combinatorial one?

Binomial coefficients everywhere

Counting distributions of balls

The basic problem

Given a set m objects and n cells (boxes, bins, etc.), how many ways can they be distributed?

Counting distributions of balls

The basic problem

Given a set m objects and n cells (boxes, bins, etc.), how many ways can they be distributed?

We can make the problem more interesting by adding constraints

- Identical/distinct objects
- Identical/distinct cells
- Empty/non-empty cells allowed
- Upper and/or lower bounds for number of objects in a cell

Bars and balls argument

Given m identical objects and n distinct cells, the number of ways to distribute the objects in the cells such that no cells are empty is

$$\binom{m-1}{n-1}$$

Bars and balls argument

Given m identical objects and n distinct cells, the number of ways to distribute the objects in the cells such that no cells are empty is

$$\binom{m-1}{n-1}$$

Example

Amanda wants to give her 3 children \$10 so everyone has > 0

There are $\binom{9}{2}$ ways to insert 2 bars in 9 gaps.

For m objects, we can insert $n-1$ bars into $m-1$ gaps.

Equivalent problem – integer composition

Problem

How many positive integer solutions for

$$x_1 + x_2 + ... x_n = m$$

Equivalent problem – integer composition

Problem

How many positive integer solutions for

$$x_1 + x_2 + \dots x_n = m$$

This is again

$$\binom{m-1}{n-1}$$

A harder problem – integer partition

What if when we distribute balls, we do not distinguish the bins?

Example

Amanda wants to divide her 10 one dollar bills into three piles. She does not care the order of the piles, e.g., these are counted as one way

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There does not exist a simple solution 😞.

A harder problem – integer partition

This is equivalent to

The integer partition problem

For fixed m and n, how many positive integer solutions for

$$a_1 + a_2 + \dots a_n = m$$

such that $a_1 \geq a_2 \geq \dots a_n$? What if we only fix m?

A harder problem – integer partition

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The integer partition problem

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Challenge

In the movie *The Man Who Knew Infinity (2015)*, G. H. Hardy says the answer for $m = 100$ with n allowed to be anything is $204,226$. Is this true?

Hardy and Ramanujan

G. H. Hardy

Srinivasa Ramanujan

Walk on the edges of a grid. Can only go Up (U) or Right (R).

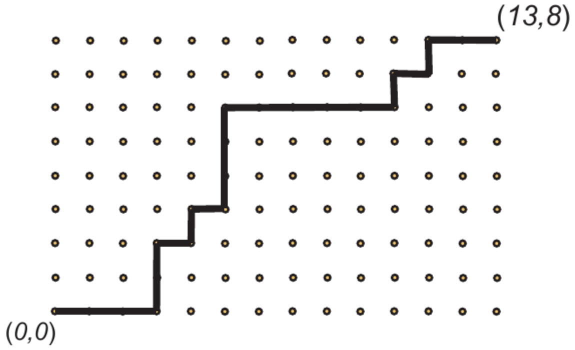


Figure 2.26: A Lattice Path

Lattice paths

The number of lattice paths from $(0,0)$ to (m,n) is $\binom{m+n}{m}$.

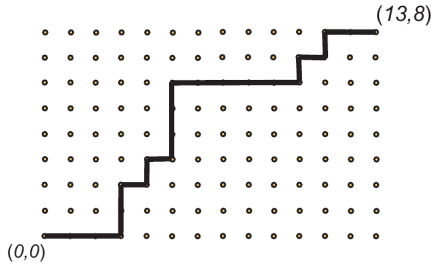


FIGURE 2.26: A LATTICE PATH

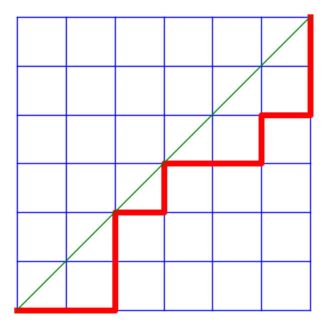
A lattice path is equivalent to a string consisting of m U's and n R's.

The above path is RRRUURURUUURRRRURURR.

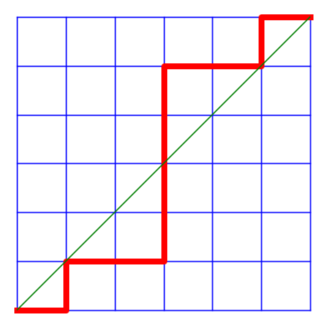
Lattice paths – Below the diagonal

Problem

How many lattice paths from $(0, 0)$ to (n, n) which does not go above the diagonal?

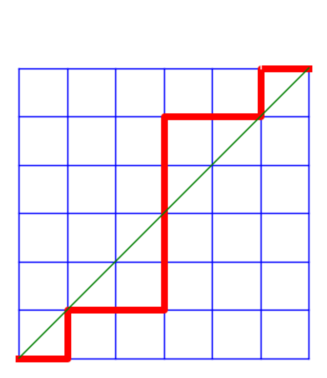


Good



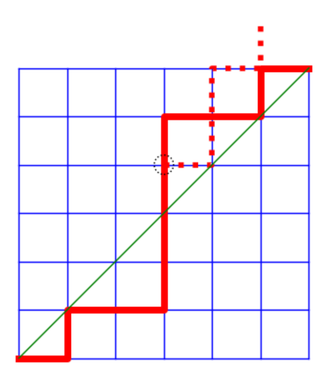
Bad

Lattice paths – Below the diagonal



The bad path must first cross the diagonal and arrive at $(j, j+1)$ – crossing point.

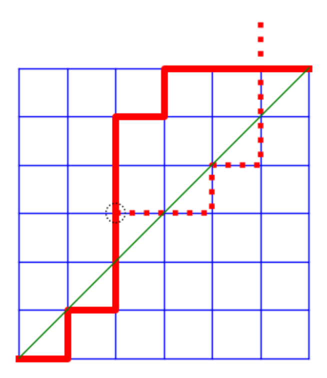
Lattice paths – Below the diagonal



The bad path must first cross the diagonal and arrive at $(j, j+1)$ – crossing point.

Fact Flip the path along 45 degree angle after the crossing point, the path will end at $(n-1, n+1)$.

Lattice paths – Another example of bad path



Fact Flip the path along 45 degree angle after the crossing point, the path will end at $(n-1, n+1)$.

Lattice paths – Below the diagonal

The number of bad paths = The number of paths ending at $(n-1, n+1) = \binom{2n}{n-1}$.

The total number of paths ending at (n, n) is $\binom{2n}{n}$.

Lattice paths – Below the diagonal

The number of bad paths = The number of paths ending at $(n-1, n+1) = \binom{2n}{n-1}$.

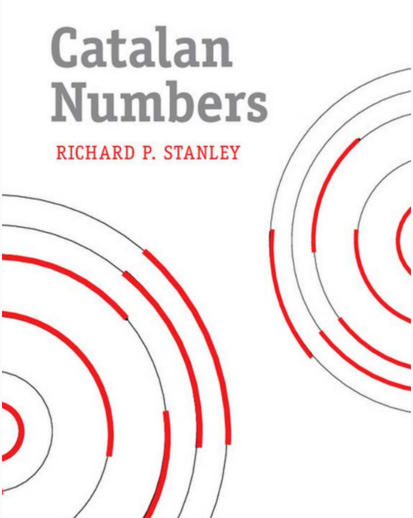
The total number of paths ending at (n, n) is $\binom{2n}{n}$.

The answer

So the number good path is the Catalan Number

$$C(n) = \binom{2n}{n} - \binom{2n}{n-1} = \frac{\binom{2n}{n}}{n+1}.$$

Often appears in combinatorics. Stanely's book gives 214 interpretations.



Binomial Theorem

An observation

Does this remind you of something?

$$
\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2 + 2\,ab + b^2 \\
(a+b)^3 &= a^3 + 3\,a^2b + 3\,ab^2 + b^3 \\
(a+b)^4 &= a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4 \\
(a+b)^5 &= a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5
\end{aligned}
$$

Pascal's triangle

Compare with this

								1								
							1		1							
						1		2		1						
					1		3		3		1					
				1		4		6		4		1				
			1		5		10		10		5		1			
		1		6		15		20		15		6		1		
	1		7		21		35		35		21		7		1	
1		8		28		56		70		56		28		8		1

Table 3.2: Pascal's Triangle

Binomial theorem

Theorem 2.30

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Explanation

$$(x+y)^n = (x+y)(x+y)(x+y)\ldots(x+y)$$

When we expand it, from each of the n terms, we choose either x or y. If x is chosen k times, then y is chosen $n-k$ times.

Applying the binomial theorem

Theorem 2.30

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Quiz

What is the coefficient of $a^{14}b^{18}$ in $(3a^2 - 5b)^{25}$?

Applying the binomial theorem

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Quiz

What is the coefficient of $a^{14}b^{18}$ in $(3a^2 - 5b)^{25}$?

Answer

$$[(a^2)^7 b^{18}](3a^2 - 5b)^{25}$$

$$= [(a^2)^7 b^{18}] \sum_{k=0}^{25} \binom{25}{k} (3a^2)^k (-5b)^{25-k} = \binom{25}{7} (3)^7 (-5)^{18}$$

Applying the binomial theorem

Theorem 2.30

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Quiz – a new proof?

$$C(n,0) + C(n,1) + \cdots + C(n,n) = 2^n.$$

Applying the binomial theorem

Theorem 2.30

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Quiz – a new proof?

$$C(n,0) + C(n,1) + \cdots + C(n,n) = 2^n.$$

Proof

$$= C(n,0)1^0 1^n + C(n,1)1^1 1^{n-1} + \cdots + C(n,n)1^n 1^0$$
$$= (1+1)^n = 2^n.$$

Applying the binomial theorem

Theorem 2.30

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Quiz – a new proof?

$$C(n,0)2^0 + C(n,1)2^1 + \cdots + C(n,n)2^n = 3^n.$$

Applying the binomial theorem

Theorem 2.30

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Quiz – a new proof?

$$C(n,0)2^0 + C(n,1)2^1 + \cdots + C(n,n)2^n = 3^n.$$

Proof

$$\begin{aligned} &C(n,0)2^0 + C(n,1)2^1 + \cdots + C(n,n)2^n \\ &= C(n,0)2^0 1^n + C(n,1)2^1 1^{n-1} + \cdots + C(n,n)2^n 1^0 \\ &= (1+2)^n = 3^n. \end{aligned}$$

Multinomial Coefficients

Multinomial coefficients

We have n distinct balls.

If we paint k of them red and the other blue, there are $\binom{n}{k}$ choices.

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If we pain k_1 balls red, k_2 blue, the rest $k_3 = n - k_1 - k_2$ purple, there are

$$\binom{n}{k_1}\binom{n-k_1}{k_2} = \frac{n!}{k_1!(n-k_1)!}\frac{(n-k_1)!}{n-(k_1+k_2)!} = \frac{n}{k_1!k_2!k_3!}$$

choices.

Multinomial coefficients

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choices.

This is called the multinomial coefficients and write

$$\binom{n}{k_1, k_2, k_3, \dots, k_r} = \frac{n!}{k_1!k_2!k_3!\dots!k_r!}.$$

Generalize binomial coefficients.

Multinomial Theorem

Theorem 2.33

$$(x_1 + x_2 + \dots + x_r)^n = \sum_{k_1+k_2+\dots+k_r=n} \binom{n}{k_1, k_2, \dots, k_r} x_1^{k_1} x_2^{k_2} \dots x_r^{k_r}$$

Multinomial Theorem

Theorem 2.33

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{k_1+k_2+\cdots+k_r=n} \binom{n}{k_1, k_2, \ldots, k_r} x_1^{k_1} x_2^{k_2} \ldots x_r^{k_r}$$

Quiz

What is the coefficient of $a^6 b^8 c^6 d^6$ in

$$(4a^3 - 5b + 9c^2 + 7d)^{19}$$

Multinomial Theorem

Theorem 2.33

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{k_1+k_2+\cdots+k_r=n} \binom{n}{k_1, k_2, \ldots, k_r} x_1^{k_1} x_2^{k_2} \ldots x_r^{k_r}$$

Quiz

What is the coefficient of $a^6 b^8 c^6 d^6$ in

$$(4a^3 - 5b + 9c^2 + 7d)^{19}$$

It is $\binom{19}{2,8,3,6} 4^2 (-5)^8 9^3 7^6$.

Special topic – Catalan numbers (Part 1)

Catalan number and Dyck path

A Dyck path of length $2n$ is a lattice path in $\mathbb{Z}^2$ from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$, with the additional condition that the path never passes below the x-axis.

Catalan number and Dyck path

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Quiz

What is $C(1), C(2), C(3), C(4)$? The number of Dyck paths of length 2, 4, 6, 8?

Catalan number and Dyck path

A Dyck path of length $2n$ is a lattice path in $\mathbb{Z}^2$ from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$, with the additional condition that the path never passes below the x-axis.

Quiz

What is $C(1), C(2), C(3), C(4)$? The number of Dyck paths of length 2, 4, 6, 8?

Both are $1, 2, 5, 14$. Coincidence?

Catalan number and Dyck path

Quiz

Why does the number of Dyck paths of length $2n$ equal $C(n)$?

Catalan number and Dyck path

Quiz

Why does the number of Dyck paths of length $2n$ equal $C(n)$?

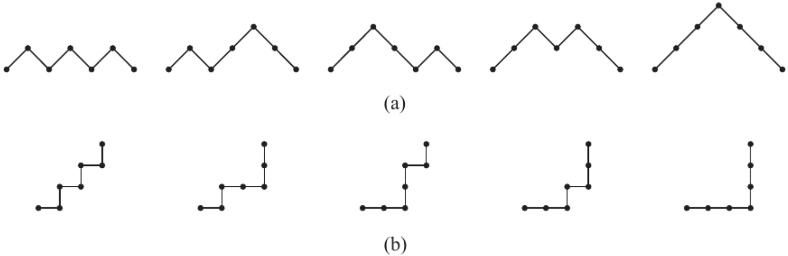

Figure 1.5. The five Dyck paths of length six.

Triangulation

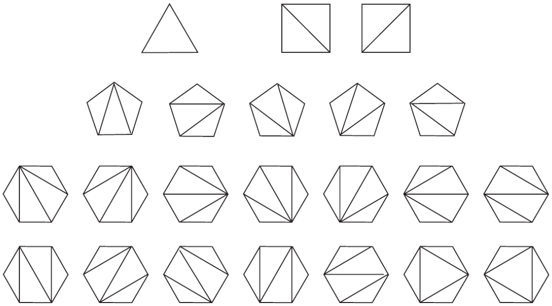

Figure 1.1. Triangulated polygons.

A *triangulation* of a convex polygon with $n+2$ vertices is set of $n-1$ diagonals which do not cross.

Catalan number and triangulation

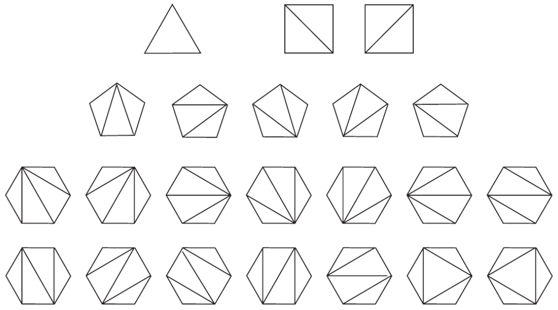

Figure 1.1. Triangulated polygons.

Quiz

The numbers of triangulations of heptagon (7-gon) is?

Catalan number and triangulation

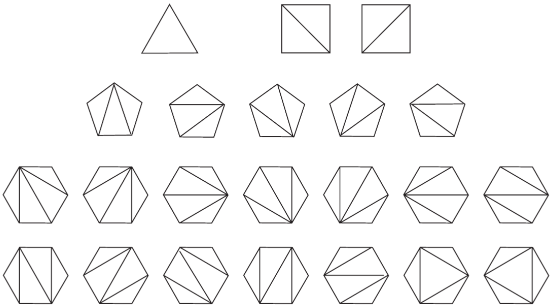

Figure 1.1. Triangulated polygons.

Quiz

The numbers of triangulations of heptagon (7-gon) is?

Claim

The number of triangulation of a convex $n + 2$-gon is $C(n)$.

Catalan number and binary trees

In a binary tree each node has either no children, one left-child, one right-child, or two children.

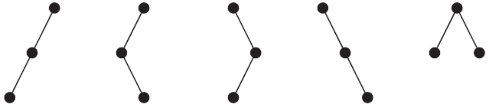

Figure 1.3. The five binary trees with three vertices.

Quiz

How many binary tree of $1, 2, 3$ and 4 nodes?

Catalan number and binary trees

In a binary tree each node has either no children, one left-child, one right-child, or two children.

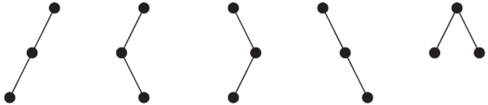

Figure 1.3. The five binary trees with three vertices.

Quiz

How many binary tree of $1, 2, 3$ and 4 nodes?

Claim

The number of binary trees of n nodes is $C(n)$.

Plane trees

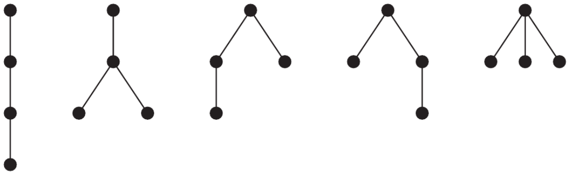

Figure 1.4. The five plane trees with four vertices.

Formal definition

A plane tree consists of a root node an ordered list of plane trees (subtrees).

Catalan number and plane trees

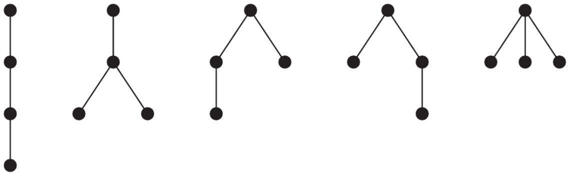

Figure 1.4. The five plane trees with four vertices.

Quiz

Let G_n be the number of plane trees of n nodes. What is G_5?

Catalan number and plane trees

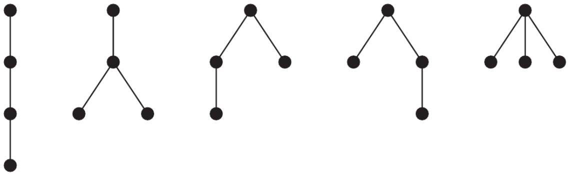

Figure 1.4. The five plane trees with four vertices.

Quiz

Let G_n be the number of plane trees of n nodes. What is G_5?

Claim

$$G_{n+1} = C_n$$

Catalan Number everywhere

1.5.1 Theorem. *The Catalan number C_n counts the following:*

(i) *Triangulations T of a convex polygon with $n+2$ vertices.*
(ii) *Binary trees B with n vertices.*
(iii) *Plane trees P with $n+1$ vertices.*
(iv) *Ballot sequences of length $2n$.*
(v) *Parenthesizations (or bracketings) of a string of $n+1$ x's subject to a nonassociative binary operation.*
(vi) *Dyck paths of length $2n$.*

We will prove some of these in the coming days.

Appendix

Self-study guide (for people who missed the class)

- Read textbook 2.5-2.8.
- Watch online video lectures 1 to 7 here.
- Recommended exercises Have a quick look of
  - Textbook 2.9, 15–33 (no solution book available)
  - Online exercises here, 15-20 (solutions on the web page)