3 – Recursion and Induction

Combinatorics 1M020

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The Meaning of Statements

What’s the next number?

It is pretty easy to guess what is the next number

- 2, 5, 8, 11, 14, 17, 20, 23, 26, ...
- 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, ...
- 1, 2, 5, 14, 42, 132, 429, 1430, 4862, ...
- 2, 6, 12, 20, 30, 42, 56, 72, 90, 110, ...
- 2, 3, 6, 11, 18, 27, 38, 51, ...

Quiz

What is your guess?

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- 2, 6, 12, 20, 30, 42, 56, 72, 90, 110, ...
- 2, 3, 6, 11, 18, 27, 38, 51, ...

Quiz

What is your guess?

You do no have to guess. Try OEIS – Online Encyclopedia of Integer Sequences!

What does the dots mean?

What about 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, 2, 3, 4, 5, 2, 3, 4, 5, 6, 2, 3, 4, 5, 6, 1, 2, 3, 4, 5, 2, 3, 4, 5, 6, ...?

What is the precise meaning of

$$1 + 2 + 3 + \cdots + 6$$

Sum of the first integers? Or sum of the first 19 terms of the above sequence?

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What is the precise meaning of

$$1 + 2 + 3 + \cdots + 6$$

Sum of the first integers? Or sum of the first 19 terms of the above sequence?

Challenge

Can you find what the sequence is?

Make definition precise – summation

Let $f : \mathbb{N} \to \mathbb{N}$ be a function. Let

$$\sum_{i=1}^{1} f(i) = f(1)$$

and for $n > 1$

$$\sum_{i=1}^{n} f(i) = f(n) + \sum_{i=1}^{n-1} f(i)$$

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Or more concisely

$$\sum_{i=1}^{n} f(i) = \begin{cases} f(1) & n = 1 \\ f(n) + \sum_{i=1}^{n-1} f(i) & n > 1 \end{cases}$$

Make definition precise – factorial

Previously, we defined

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$$

With new notation, now we write

$$n! = \begin{cases} 1 & n = 1 \\ n \times (n-1)! & n > 1 \end{cases}$$

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Note When it is clear from the context, it is Okay to use the .... Professional mathematicians do that too.

Recursive definition in Python

```
def sumrecursive(n):
    if n == 1:
        return 2;
    else :
        return sumrecursive(n-1) + (n*n - 2*n + 3)
sumrecursive(3)
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The output of this code is 11. What about sumrecursive(4)

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Quiz

The output of this code is 11. What about `sumrecursive(4)`

`sumrecursive(n)` simply computes

$$2 + 3 + 6 + 11 + 18 + 27 + 38 + 51 + \cdots + (n^2 - 2n + 3)$$

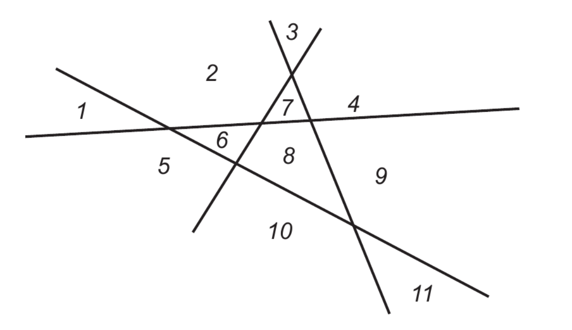
Solving Combinatorial Problems Recursively

Lines and areas

Each pair of the 4 lines intersects.

No point in the plane belongs to more than two lines.

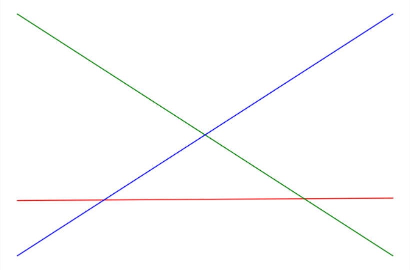
These 4 lines determine 11 regions.



Lines and areas–Recursion

Let n be the number of lines and $r(n)$ be the number of regions. Then

$$r(n) = n + r(n-1)$$



Explanation: the n-th line is divided into n segments.

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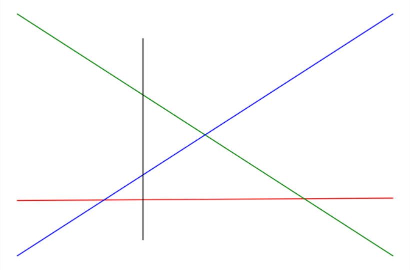


Explanation: the n-th line is divided into n segments.

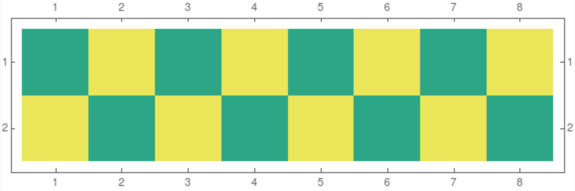
Lines and areas–Recursion

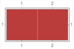
Problem

Can you find $r(n)$?

$$\begin{aligned} r(n) &= n + r(n-1) \\ &= n + (n-1) + \cdots + 2 + r(1) \\ &= n + (n-1) + \cdots + 2 + 2 \\ &= \frac{n(n+1)}{2} + 1 \end{aligned}$$

Tiling chessboard



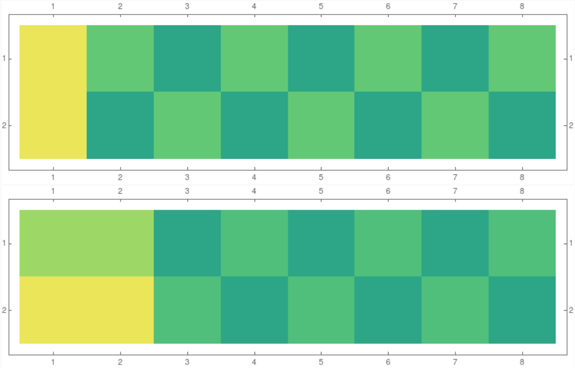
A $2 \times n$ chess board needs to be titled with 2×1 dominoes .
Let $t(n)$ be the number of possible ways.

Quiz

Why for $n \geq 2$

$$t(n) = t(n-1) + t(n-2)$$

Tiling chessboard



There are two possible ways to start

- One vertical domino, which left $t(n-1)$ choices for the rest.
- Two horizontal dominoes, which left $t(n-2)$ choices for the rest.

Ternary string

A ternary string of alphabet $\{a, b, c\}$ is good if there's no c followed by a. Example

- a a b c b b a – good
- c a b b a a c b c c – bad

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Let $g(n)$ be the number of good string of length n.

Quiz

What is $g(1)$ and $g(2)$?

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Quiz

What is $g(1)$ and $g(2)$?

claim

$$g(n) = 3g(n-1) - g(n-2).$$

Ternary string

There are three ways to construct a good string of length n –

- Good string of length $n - 1$ + "c" – 😃
- Good string of length $n - 1$ + "b" – 😃
- Good string of length $n - 1$ + "a" – 😒 problem here

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  - Everything else – 😃

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So in total there are $3g(n-1) - g(n-2)$ good strings of length n.

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  - Everything else – 😃

So in total there are $3g(n-1) - g(n-2)$ good strings of length n.

Challenge Can you find $g(6)$?

Find the Greatest Common Divisors

Division Theorem

Theorem 3.7 (Division Theorem)**.** *Let m and n be positive integers. Then there exist unique integers q and r so that*

$$m = q \cdot n + r \quad \textit{and} \quad 0 \leq r < n.$$

We call q the ***quotient*** *and r the* ***remainder****.*

Seems obvious? Not so quick. Easy to prove uniqueness. But how do we know such q and r exists?

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Proof by contradiction

Let t be the least positive integer for which there are integers m and n with $m + n = t$, but there do not exist integers q and r satisfying the theorem. Show such t cannot exist.

Division Theorem – details

Assumption: $t = m + n$ is the least t such that no q and $0 \leq r \leq n$ makes $m = q \cdot n + r$.

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Find contradiction as follows

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- $n \neq 1$, otherwise take $q = m$ and $r = 0$.
- $m \neq 1$, otherwise take $q = 0$ and $r = 1$.
- There exists q and r such that

$$m - 1 = q \times n + r \quad \text{and} \quad 0 \leq r < n.$$

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- If $r + 1 < n$, then contradiction.
- If $r + 1 = n$, then

$$m = q \times n + n = (q + 1)n + 0$$

Euclidean Algorithm

Theorem 3.8 (Euclidean Algorithm). *Let m, n be positive integers with $m > n$ and let q and r be the unique integers for which*

$$m = q \cdot n + r \quad \text{and} \quad 0 \leq r < n.$$

If $r > 0$, then $\gcd(m, n) = \gcd(n, r)$. If $r = 0$, then n divides m, and $\gcd(m, n) = n$.

This gives us an algorithm to compute the greatest common divisor

```
def gcd(m, n):
    if m % n == 0:
        return n
    else:
        return gcd(n ,m%n)
gcd(12 ,5)
```

Euclidean Algorithm – history

The algorithm appears in Euclid's Elements (300 BC).
The algorithm was probably not discovered by Euclid.
He compiled results from earlier mathematicians in his Elements.
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"Everything of importance has been said before by somebody who did not discover it" — Alfred North Whitehead.

Induction

Let S_n be an open statement involving a positive integer n. If S_1 is true, and if for each positive integer k, assuming that the statement S_k is true implies that the statement S_{k+1} is true, then S_n is true for every positive integer n.

Example – sum of the first n integers

How to prove by induction

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

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- Check $\frac{1(1+1)}{2} = 1$. So holds for $n = 1$.

Example – sum of the first n integers

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$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

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- Then

$$\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} + (k+1) = \frac{(k+2)(k+1)}{2}$$

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- Assume that for some positive integer k

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- Then

$$\sum_{i=1}^{k+1}(2i-1) = k^2 + (2k+1) = (k+1)^2$$

Example – sum of $1/\sqrt{n}$

Problem

Let $S_n = 1/\sqrt{1} + 1/\sqrt{2} + \cdots + 1/\sqrt{n}$. Show that for $n \geq 2$,

$$S_n > \sqrt{n}$$

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- Check $n = 2$.
- Assume that for some positive integer $k \geq 2$ $S_k > \sqrt{k}$
- Then $\sqrt{k+1} > S_{k+1}$ because

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Challenge Show that $S_n < 2\sqrt{n}$ and $2\sqrt{n} - S_n < 3/2$.

Example – sum of binomial coefficients

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$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}, \qquad (n \geq k \geq 0)$$

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- Then

$$\sum_{i=k+1}^{m+1} \binom{i}{k} = \binom{m+1}{k+1} + \binom{m+1}{k} = \binom{m+2}{k+1}$$

Strong induction

Let $f(1) = 3$, $f(2) = 5$, and $f(n) = 2f(n-1) - f(n-2)$.

Quiz

If we compute the first numbers, we get the sequence

$$3, 5, 7, 9, 11, \ldots$$

Do you see a pattern?

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- We have made assumptions for $f(k)$, not $f(k-1)$.

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How to prove $f(n) = 2n + 1$? (The correct way)

- Check holds for $n = 1$ and $n = 2$.

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$$f(k+1) = 2f(k) - f(k-1) = 2(2k+1) - (2(k-1)+1) = 2(k+1)+1.$$

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- Then

$$f(k+1) = 2f(k) - f(k-1) = 2(2k+1) - (2(k-1)+1) = 2(k+1)+1.$$

- We use assumptions for both $f(k)$ and $f(k-1)$.

Special topic – Catalan numbers (Part 2)

Catalan number

Recall that we have defined Catalan number $C(n) = \binom{2n}{n}\frac{1}{n+1}$.

$C(n)$ is the number of Dyck path of length $2n$.

$C(n)$ is also "good" lattice walks from $(0, 0)$ to (n, n).

$C(0) = 1$, $C(1) = 1$, $C(2) = 2$, $C(3) = 5$, $C(4) = 14$.

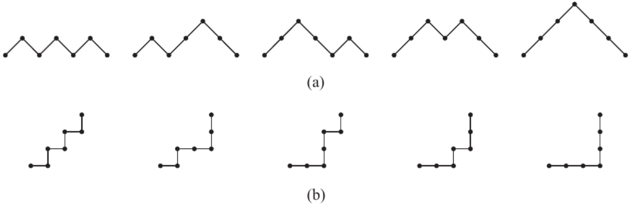
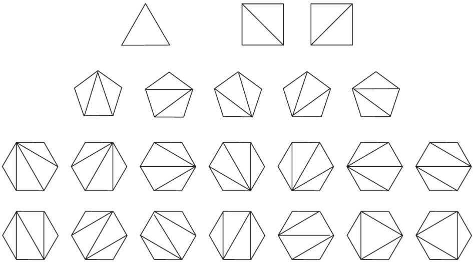


Figure 1.5. The five Dyck paths of length six.

Catalan number and triangulation

A *triangulation* of a convex polygon with $n + 2$ vertices is set of $n - 1$ diagonals which do not cross.



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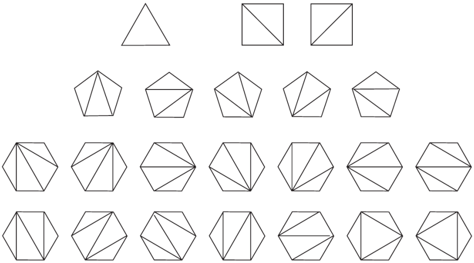


Figure 1.1. Triangulated polygons.

Let T_n be the number of triangulation of a convex n-gon. Then

$$T_{n+2} = C(n) = \frac{1}{n+1}\binom{2n}{n}, \qquad (n \geq 0).$$

Catalan number and triangulation

Claim

$$T_{n+3} = \sum_{k=0}^{n} T_{k+2} T_{n-k+2}$$

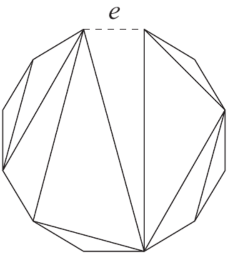


Figure 1.2. The recursive structure of a triangulated polygon.

Catalan number and triangulation

Claim

$$T_{n+3} = \sum_{k=0}^{n} T_{k+2} T_{n-k+2}$$

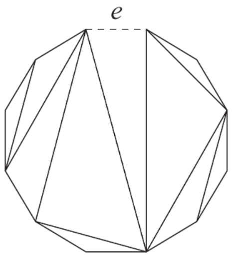


Figure 1.2. The recursive structure of a triangulated polygon.

Fix an edge e in a $n+3$-gon. Removing it gives two triangulated polygon with $k+2$ and $n-k+2$ vertices.

Catalan number and triangulation

Since

$$T_{n+3} = \sum_{k=0}^{n} T_{k+2} T_{n-k+2}$$

If

$$C(n+1) = \sum_{k=0}^{n} C(k) C(n-k).$$

then

$$T_{n+2} = C(n)$$

Catalan number and Dyck path

Recall that the number of Dyck paths of length $2n$ is $C(n)$.

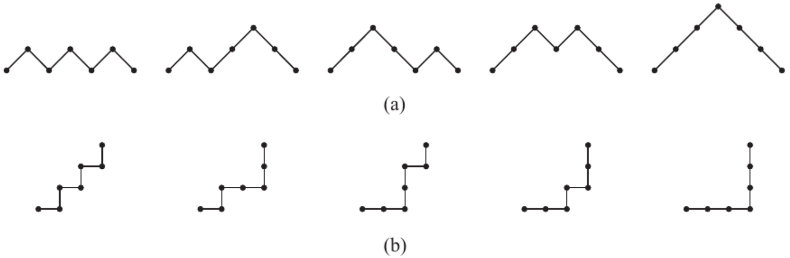
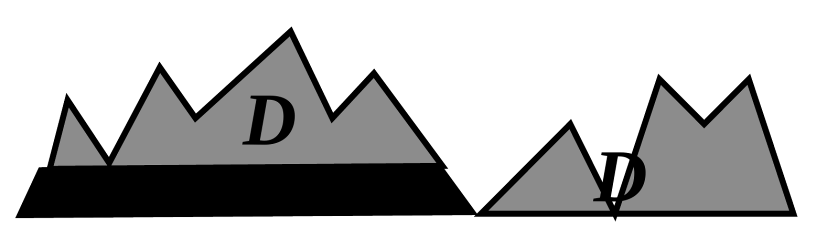


Figure 1.5. The five Dyck paths of length six.

Catalan number and Dyck path – Decomposition



A Dyck path of length $2(n+1)$ can be decomposed into two Dyck paths of length $2k$ and $2(n-k)$, where $2(k+1)$ is the first time the path comes back 0 again.

Thus

$$C(n+1) = \sum_{k=0}^{n} C(k)C(n-k).$$

Catalan number and binary trees

In a binary tree each node has either no children, one left-child, one right-child, or two children.

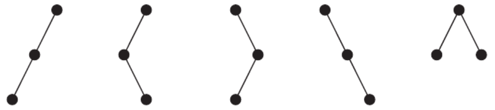


Figure 1.3. The five binary trees with three vertices.

Claim

The number of binary trees of n nodes is $C(n)$.

Catalan number and binary trees – Proof 1

Let B_n be the number of binary trees of n nodes.

Quiz

Can you see why

$$B_{n+1} = \sum_{k=0}^{n} B_k B_{n-k}.$$

Catalan number and binary trees – Proof 1

Let B_n be the number of binary trees of n nodes.

Quiz

Can you see why

$$B_{n+1} = \sum_{k=0}^{n} B_k B_{n-k}.$$

A binary tree of size $n+1$ can be decomposed of left subtree of size k and a right subtree of $n-k$.

Catalan number and binary trees – Proof 2

A bijection between binary trees and triangulation

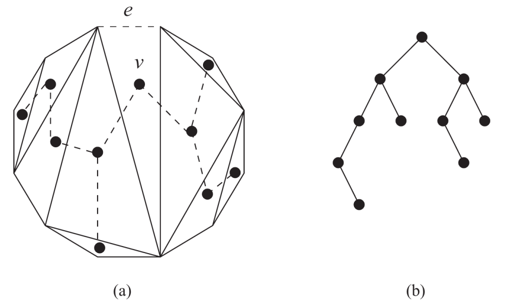


Figure 1.6. A binary tree associated with a triangulated polygon.

Plane trees

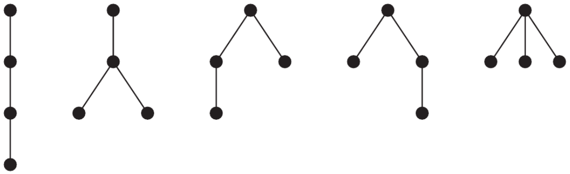


Figure 1.4. The five plane trees with four vertices.

Formal definition

A plane tree consists of a root node an ordered list of plane trees (subtrees).

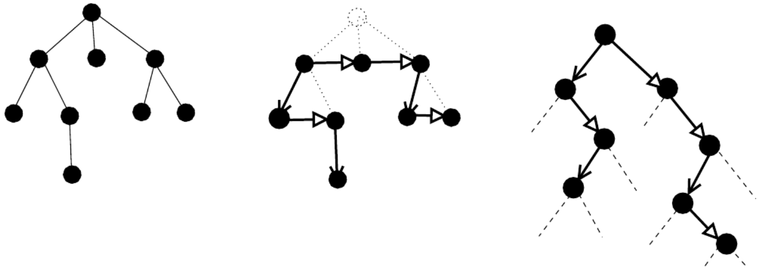
Catalan number and plane trees

Claim

Let G_n be the number of plane trees of n nodes.

$$G_{n+1} = C(n)$$

A bijection between general trees and binary trees. $G_{n+1} = B_n$.



Appendix

Self-study guide (for people who missed the class)

Watch online video lectures 1 to 7 here.

- Read textbook chapter 3, Inductive Proof – Common Errors and Pitfalls and also these slides.
- Watch online video lectures 8 to 9 here and all video lectures here.
- Recommended exercises Have a quick look of
  - Textbook 3.11 (some solutions here)