8 – Generating Functions Part (1)

Combinatorics 1M020

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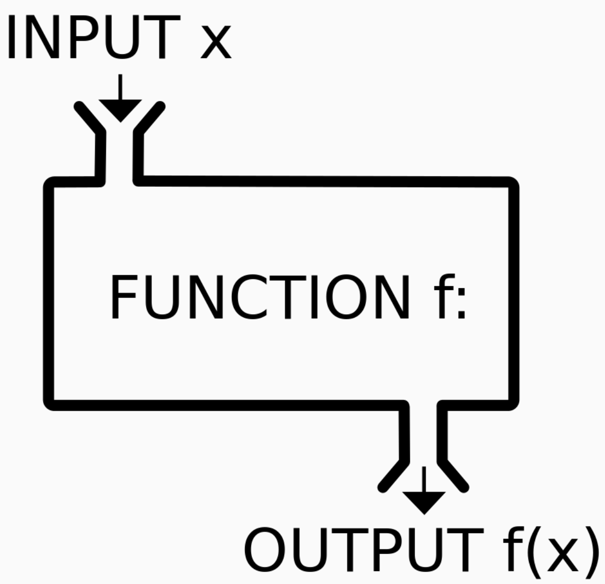
Basic notion and terminology

Review – What is a function?

A function can be seen a rule to convert input to output. (Just like a function in computer languages)



What is a GF (generating function)

Given an infinite sequence $\sigma = (a_0, a_1, \dots)$, we associate it with a "function" $F(x)$ written as

$$F(x) = \sum_{n \geq 0} a_n x^n,$$

called the generating function of σ.

Warning ⚠

Very formally speaking, $F(x)$ is not a function and we do not care if the sum converges.

We just pretend in this class that they are well-defined functions.

There are GFs that do not correspond to any function, e.g.,

$$\sum_{n \geq 0} n! x^n.$$

Examples of GF

If $a_0 = 1$ and $a_n = 0$ for all $n \geq 1$, then

$$F(x) = \sum_{n \geq 0} a_n x^n = 1.$$

If $a_n = 1$ for all $n = 0, \ldots, 4$, and $a_n = 0$ for all $n \geq 5$ then

$$F(x) = 1 + x + x^2 + x^3 + x^4 = \frac{1 - x^5}{1 - x}$$

If $a_n = 1$ for all $n = 0, \ldots, m$, and $a_n = 0$ for all $n \geq m + 1$ then

$$F(x) = 1 + x + x^2 + \cdots + x^m = \frac{1 - x^{m+1}}{1 - x}$$

Examples of GF

If $a_n = 1$ for all $n \geq 0$, then

$$F(x) = \sum_{n \geq 0} x^n = \frac{1}{1 - x}.$$

If $a_n = 1/n!$ for all $n \geq 0$, then

$$F(x) = \sum_{n \geq 0} \frac{1}{n!} x^n = e^x.$$

If $a_0 = 0$ and $a_n = 1/n$ for all $n \geq 1$, then

$$\sum_{n \geq 1} \frac{1}{n} x^n = -\log\left(\frac{1}{1 - x}\right)$$

Examples of GF

Quiz

$$\sum_{n\geq 0} \binom{m}{n} x^n = ?$$

Quiz

$$\sum_{n\geq 0} \frac{1}{(2n)!} x^{2n} = ?$$

Generating functions and combinatorics

Combinatorial Sum

Let $\mathcal{A}$ and $\mathcal{B}$ be two sets contains objects which has sizes, e.g.,

$\mathcal{A} = \{$💵, 💵💵, 💵💵💵, ...$\}$,
$\mathcal{B} = \{$🍎, 🍐, 🍎🍎, 🍐🍐, 🍎🍎🍎, 🍐🍐🍐, ...$\}$.
$\mathcal{C} = \mathcal{A} \cup \mathcal{B} = \{$🍎, 🍐, 💵, 🍎🍎, 🍐🍐, 💵💵, 🍎🍎🍎, 🍐🍐🍐, 💵💵💵 ...$\}$.

Let a_n (b_n) be the number of objects of size n in $\mathcal{A}$ ($\mathcal{B}$).

Let $A(x)$ and $B(x)$ be the GFs for a_n and b_n.

Punch line! – Then the GF for $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ is

$$C(x) = A(x) + B(x) = \sum_{n \geq 0} a_n x^n + \sum_{n \geq 0} b_n x^n = \sum_{n \geq 0} (a_n + b_n) x^n$$

Combinatorial Sum – Example

$\mathcal{A} = \{$💵, 💵💵, 💵💵💵, ...$\}$,
$\mathcal{B} = \{$🍎, 🍐, 🍎🍎, 🍐🍐, 🍎🍎🍎, 🍐🍐🍐, ...$\}$.
$\mathcal{C} = \mathcal{A} \cup \mathcal{B} = \{$🍎, 🍐, 💵, 🍎🍎, 🍐🍐, 💵💵, 🍎🍎🍎, 🍐🍐🍐, 💵💵💵 ...$\}$.

In the example above, $a_n = 1$ and $b_n = 2$ for all n. So

$$A(x) = \sum_{n \geq 1} x^n = \frac{x}{1-x},$$

$$B(x) = \sum_{n \geq 1} 2x^n = 2\frac{x}{1-x},$$

$$C(x) = A(x) + B(x) = 3\frac{x}{1-x} = \sum_{n \geq 1} 3x^n.$$

Combinatorial Product

Let $\mathcal{A}$ and $\mathcal{B}$ be two sets contains objects which has sizes, e.g.,

$\mathcal{A} = \{\emptyset,$ 💵, 💵💵, 💵💵💵, $\dots\}, \quad \mathcal{B} = \{\emptyset,$ 🍎, 🍎🍎, 🍎🍎🍎, $\dots\}.$

$\mathcal{C} = \mathcal{A} \times \mathcal{B} = \{(\emptyset, \emptyset), (\emptyset,$ 🍎$),($💵$, \emptyset), (\emptyset,$ 🍎🍎$), ($💵💵$, \emptyset), ($💵, 🍎$), \dots\}.$

Let a_n (b_n) be the number of objects of size n in $\mathcal{A}$ ($\mathcal{B}$).

Let $A(x)$ and $B(x)$ be the GFs for a_n and b_n.

Punch line! – Then the GF for $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ is

$$
\begin{aligned}
C(x) = A(x)B(x) &= \left(\sum_{n \geq 0} a_n x^n\right)\left(\sum_{n \geq 0} b_n x^n\right) \\
&= \sum_{n \geq 0} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n
\end{aligned}
$$

Newton's Binomial Theorem

Extend the definition of $\binom{r}{k}$

For integers $n \geq m \geq 0$, we have defined that

$$\binom{n}{m} = \frac{P(n,m)}{m!} = \frac{n(n-1)(n-2)\dots(n-m+1)}{m!}$$

But let's be crazier and let $r \in \mathbb{R}$ and $k \in \mathbb{Z}$ and define

$$\binom{r}{k} = \begin{cases} \frac{r(r-1)(r-2)\dots(r-k+1)}{k!} & k > 0, \\ 1 & k = 0, \\ 0 & k < 0. \end{cases}$$

Extend the definition of $\binom{r}{k}$

$$\binom{r}{k} = \begin{cases} \frac{r(r-1)(r-2)\dots(r-k+1)}{k!} & k > 0, \\ 1 & k = 0, \\ 0 & k < 0. \end{cases}$$

For integers $k > r \geq 0$, $\binom{r}{k} = 0$ – there is no way to choose 5 🍌 out of 3 🍌.

For integers $r \geq 0$ and $k < 0$, $\binom{r}{k} = 0$ – there is no way to choose -1 🍌 out of 3 🍌.

If $k = 0$, $\binom{r}{0} = 1$ – there is one way to choose 0 🍌 (do nothing)

Example

$$\binom{-7/2}{5} = \frac{(-7/2)(-9/2)(-11/2)(-13/2)(-15/2)}{5!}.$$

Newton's Binomial Theorem

Quiz

How is 2^π defined?

Theorem 8.10

For all $p \in \mathbb{R}$ with $p \neq 0$,

$$(1+x)^p = \sum_{n \geq 0} \binom{p}{n} x^n$$

When $p \in \mathbb{N}$, this is just binomial theorem.

Proof by taking the Taylor expansion of $(1+x)^p$ at $x = 0$.

Combinatorial Product – revisited

The sets

$$\mathcal{A} = \{\emptyset, \text{💵}, \text{💵💵}, \text{💵💵💵}, \dots\}, \quad \mathcal{B} = \{\emptyset, \text{🍎}, \text{🍎🍎}, \text{🍎🍎🍎}, \dots\}.$$

$$\mathcal{C} = \mathcal{A} \times \mathcal{B} = \{(\emptyset, \emptyset), (\emptyset, \text{🍎}), (\text{💵}, \emptyset), (\emptyset, \text{🍎🍎}), (\text{💵💵}, \emptyset), (\text{💵}, \text{🍎}), \dots\}.$$

The GF

$$A(x) = \frac{1}{1-x}, \quad B(x) = \frac{1}{1-x}, \quad C(x) = \left(\frac{1}{1-x}\right)^2$$

By Newton's binomial theorem

$$C(x) = \sum_{n \geq 0} \binom{-2}{n} (-x)^n$$

and

$$[x^n]C(x) = (-1)^n \frac{(-2)(-3)\dots(-2-n+1)}{n!} = \frac{(n+1)!}{n!} = n+1$$

Applying Newton's Binomial Theorem

Lemma 8.12

For all integers $m \geq 0$,

$$\binom{-1/2}{m} = (-1)^m \frac{\binom{2m}{m}}{2^{2m}}.$$

Proof by induction. Quiz What is $\binom{-1/2}{0}$?

$$\binom{-1/2}{m+1} = \frac{P(-1/2, m+1)}{(m+1)!} = \frac{P(-1/2, m)(-1/2-m)}{(m+1)m!}$$
$$= \frac{-1/2-m}{m+1}\binom{-1/2}{m} = (-1)\frac{2m+1}{2(m+1)}(-1)^m \frac{\binom{2m}{m}}{2^{2m}}$$
$$= (-1)^{m+1}\frac{1}{2^{2m}}\frac{(2m+2)(2m+1)}{(2m+2)2(m+1)}\binom{2m}{m} = (-1)^{m+1}\frac{\binom{2m+2}{m+2}}{2^{2m+2}}.$$

Applying Newton's Binomial Theorem

Theorem 8.13

$$\frac{1}{\sqrt{1-4x}} = \sum_{n \geq 0} \binom{2n}{n} x^n.$$

$$(1-4x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n}(-4x)^n$$
$$= \sum_{n=0}^{\infty} (-1)^n 2^{2n} \binom{-1/2}{n} x^n$$
$$= \sum_{n=0}^{\infty} \binom{2n}{n} x^n.$$

Does this ring a bell?

Applying Newton's Binomial Theorem

Quiz - Corollary 8.14

Use

$$\frac{1}{\sqrt{1-4x}} = \sum_{n\geq 0} \binom{2n}{n} x^n$$

to show that for all integers $n \geq 0$

$$2^{2n} = \sum_{k\geq 0} \binom{2k}{k} \binom{2n-2k}{n-k}.$$

Another look at Amanda's money

Give 1 💵 to k child

The ways to give 💵 to 1 child forms the set

$$\mathcal{A} = \{💵, 💵💵, 💵💵💵, \ldots\}$$

Since there are $a_n = 1$ way to distribute n 💵 among 1 child so the child has > 0 💵 , the GF of $\mathcal{A}$ is

$$A(x) = x + x^2 + x^3 + \cdots = \frac{x}{1-x}$$

Note $a_0 = 0$.

So $\mathcal{A}^k$ contains the ways to distribute n 💵 among k children. Its GF is

$$A(x)^k = \left(\frac{x}{1-x}\right)^k$$

Give n 💵 to 5 child

Quiz

What is the combinatorial meaning of the coefficient of x^6 in

$$(x+x^2 \ldots)(x+x^2 \ldots)(x+x^2 \ldots)(x+x^2 \ldots)(x+x^2 \ldots) = \frac{x^5}{(1-x)^5}$$

It is the number of ways to distribute 6 💵 among 5 children so everyone has > 0 💵

I.e., $\binom{5}{4} = 5$ by the balls and bars argument.

💵 🚦 💵 🚦 💵 💵 🚦 💵 🚦 💵

Give n 💵 to 5 child

Use SageMath, it is also easy to see that it is 5, and in general, the coefficient of x^{n+1} is $\binom{n}{4}$.

To get this without computer or combinatorics

$$\frac{x^5}{(1-x)^5} = \frac{x^5}{4!}\frac{d^4}{dx^4}\left(\frac{1}{1-x}\right) = \frac{x^5}{4!}\sum_{n=0}^{\infty} n(n-1)(n-2)(n-3)x^{n-4}$$
$$= \sum_{n=0}^{\infty} \frac{n(n-1)(n-2)(n-3)}{4!}x^{n+1} = \sum_{n=0}^{\infty}\binom{n}{4}x^{n+1}.$$

Give 5 💵 to 3 children allowing empty hands

How many ways can we distribute 5 💵 to 3 children, allowing a child to have 0 💵?

The trick is to use bars and balls with three extra 💵, e.g.,

💵 💵 💵 🚦 💵 💵 💵 💵 🚦 💵

This example gives the children $3, 4, 1$ 💵.

But then we just take back one 💵 from each child, so they get $2, 3, 0$ 💵.

So the answer is $\binom{8}{2}$.

Give n 💵 to k children allowing empty hands

How many ways can we distribute n 💵 to k children, allowing a child to have 0 💵?

The trick is to add k extra 💵.

Then we distribute it by inserting $k-1$ bars among the $n+k-1$ gaps.

In the end, we take back one 💵 from each child.

So there this is in total $\binom{n+k-1}{k-1}$ ways to do so.

Give n 💵 to k children allowing empty hands

$(1-x)^{-n}$ is GF for $\mathcal{A}^n$ with

$$\mathcal{A} = \{\emptyset, 💵, 💵💵, 💵💵💵, \dots\},$$

$[x^k](1-x)^{-n}$ is the number of ways to distribute n one-dollar bills to k children, allowing children to have no money. By the bars and balls argument, this is $\binom{n+k-1}{k-1}$ for $k \geq 1$. So

$$\frac{1}{(1-x)^n} = 1 + \sum_{k \geq 1} \binom{n+k-1}{k-1} x^k.$$

Or by Newton's binomial theorem

$$\frac{1}{(1-x)^n} = \sum_{k \geq 0} \binom{-n}{k} (-1)^k x^k$$

Fruit basket

Example 8.5

A grocery store is preparing holiday fruit baskets for sale. Each fruit basket will have 20 pieces of fruit in it, chosen from 🍎, 🍐, 🍊, and 🍇. How many different ways can such a basket be prepared if there must be at least one apple in a basket, a basket cannot contain more than three pears, and the number of 🍊 must be a multiple of four?

Fruit basket – Equivalent problem

Let

$\mathcal{A} = \{$🍎, 🍎🍎, 🍎🍎🍎, …$\}$, $\mathcal{P} = \{\emptyset,$ 🍐, 🍐🍐, 🍐🍐🍐$\}$,

$\mathcal{O} = \{\emptyset,$ 🍊🍊🍊🍊, 🍊🍊🍊🍊🍊🍊🍊🍊, …$\}$,

$\mathcal{G} = \{\emptyset,$ 🍇, 🍇🍇, 🍇🍇🍇, …$\}$,

What is the number of objects in $\mathcal{B} = \mathcal{A} \times \mathcal{P} \times \mathcal{O} \times \mathcal{G}$ of size 20?

Quiz

What is the GF for pears?

Fruit basket

The GFs are

$$A(x) = x + x^2 + x^3 \cdots = \frac{x}{1-x}, \quad P(x) = 1 + x + x^2 + x^3,$$
$$O(x) = 1 + x^4 + x^8 \cdots = \frac{1}{1-x^4},$$
$$G(x) = 1 + x + x^2 + x^3 \cdots = \frac{1}{1-x}.$$

The GF of $\mathcal{B}$ (baskets) is

$$B(x) = A(x)P(x)O(x)G(x)$$

Fruit basket

So the answer is

$$[x^{20}]B(x) = 210,$$

and $[x^n]B(x) = n(n+1)/2$ – pretty easy by SageMath. Without computer $B(x)$ is

$$\frac{x}{1-x}(1+x+x^2+x^3)\frac{1}{1-x^4}\frac{1}{1-x} = \frac{x}{(1-x)^2(1-x^4)}(1+x+x^2+x^3).$$

$$\frac{x}{(1-x)^3} = \frac{x}{2}\sum_{n=0}^{\infty} n(n-1)x^{n-2} = \sum_{n=0}^{\infty} \frac{n(n-1)}{2}x^{n-1}$$

Integer composition – with restriction

Example 8.6. Find the number of integer solutions to the equation

$$x_1 + x_2 + x_3 = n$$

($n \geq 0$ an integer) with $x_1 \geq 0$ even, $x_2 \geq 0$, and $0 \leq x_3 \leq 2$.

The GFs for x_1, x_2, x_3 are

$$1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2}, \quad 1 + x + x^2 \cdots = \frac{1}{1-x}, \quad 1 + x + x^2$$

The GF for the solution is

$$\frac{1 + x + x^2}{(1-x)(1-x^2)} = \frac{1 + x + x^2}{(1+x)(1-x)^2}$$

Partial fraction

We can further simplify to the form

$$\frac{1 + x + x^2}{(1+x)(1-x)^2} = \frac{A}{1+x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$$

form some constant A, B, C. This implies

$$1 + x + x^2 = A(1-x)^2 + B(1-x^2) + C(1+x)$$

Equating coefficients on terms of equal degree

$$1 = A + B + C, \quad 1 = -2A + C, \quad 1 = A - B.$$

Solving this gives

$$A = \frac{1}{4}, \quad B = \frac{-3}{4}, \quad C = \frac{3}{2}$$

Integer composition – with restriction

This simplifies to

$$\begin{aligned}&\frac{1}{4}\frac{1}{1+x}-\frac{3}{4}\frac{1}{1-x}+\frac{3}{2}\frac{1}{(1-x)^2}\\=&\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n x^n-\frac{3}{4}\sum_{n=0}^{\infty}x^n+\frac{3}{2}\sum_{n=0}^{\infty}nx^{n-1}.\end{aligned}$$

So the coefficient of x^n is

$$\frac{(-1)^n}{4}-\frac{3}{4}+\frac{3(n+1)}{2}.$$

Integer composition – without restriction

This is the GF of distributing n 💵 to any number of (distinguishable) children, so no one has 0 💵.

$$1+\frac{x}{1-x}+\left(\frac{x}{1-x}\right)^2+\left(\frac{x}{1-x}\right)^3+\cdots=\frac{1}{1-\frac{x}{1-x}}=\frac{1-x}{1-2x}.$$

Since

$$\frac{1-x}{1-2x}=\left(\sum_{n\geq 0}2^n x^n\right)-\left(\sum_{n\geq 0}2^n x^{n+1}\right),$$

the coefficient of x^n is 2^{n-1} for $n\geq 1$.

By the balls and bars argument, this is 2^{n-1} for $n\geq 1$. Why?

Appendix

Self-study guide (for people who missed the class)

- Read textbook 8.1–8.4
- Watch online video lectures here.
- Recommended exercises Have a quick look of
  - Textbook 8.9, 1–19 (Some solutions here)