8 – Generating Functions Part (2)

Combinatorics 1M020

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Integer Partition

Integer partition – money definition

Example

Amanda wants to divide her 10 one dollar bills into any number of piles. She does not care the order of the piles, e.g., these are counted as one way

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How many ways could she do it?

Integer partition – formal definition

This is equivalent to

The integer partition problem

For $m \in \mathbb{N}$, let p_m be the number of positive integer solutions for

$$a_1 + a_2 + \dots a_n = m$$

such that $a_1 \geq a_2 \geq \dots a_n$, with n allowed to be any integer.

Challenge

In the movie *The Man Who Knew Infinity (2015)*, G. H. Hardy says $p_{100} = 204,226$. Is this true?

Integer partition – 8

8 distinct parts	4+1+1+1+1
7+1 distinct parts, odd parts	3+3+2
6+2 distinct parts	3+3+1+1 odd parts
6+1+1	3+2+2+1
5+3 distinct parts, odd parts	3+2+1+1+1
5+2+1 distinct parts	3+1+1+1+1+1 odd parts
5+1+1+1 odd parts	2+2+2+2
4+4	2+2+2+1+1
4+3+1 distinct parts	2+2+1+1+1+1
4+2+2	2+1+1+1+1+1+1
4+2+1+1	1+1+1+1+1+1+1+1 odd parts

Figure 8.15: The partitions of 8, noting those into distinct parts and those into odd parts.

Integer partition – GF

Quiz – What are the GFs of

$\mathcal{P}_1 = \{\emptyset,$ 🍎, {🍎, 🍎}, {🍎, 🍎, 🍎} … } ,

$\mathcal{P}_2 = \{\emptyset,$ 🍎🍎, {🍎🍎, 🍎🍎}, {🍎🍎, 🍎🍎, 🍎🍎}, … } ,

$\mathcal{P}_3 = \{\emptyset,$ 🍎🍎🍎, {🍎🍎🍎, 🍎🍎🍎}, {🍎🍎🍎, 🍎🍎🍎, 🍎🍎🍎}, … } ,

The GFs are

$$P_1(x) = \frac{1}{1-x}, P_2(x) = \frac{1}{1-x^2}, \dots, P_k(x) = \frac{1}{1-x^k}, \dots$$

Integer partition – GF

Since the set of integer partitions is $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \times \mathcal{P}_3 \dots$, the GF for $\mathcal{P}$ is

$$P(x) = \prod_{m \geq 1} P_m(x) = \prod_{m \geq 1} \frac{1}{1 - x^m}.$$

This answer of Hardy's problem

$$p_{100} = [x^{100}]P(x) = 190569292.$$

The movie lied!

Integer partition – with restriction

Quiz

What is the GF for partitions into parts ≥ 3?

$$P_{\geq 3}(x) = \prod_{m \geq 3} \frac{1}{1 - x^m}$$

Quiz

What is the GF for partitions into only odd parts?

$$O(x) = \prod_{m \geq 1} \frac{1}{1 - x^{2m-1}}$$

Integer partition – with restriction

Quiz

What is the GF for partitions into distinct parts?

The distinct part only partition is the product of

$$\mathcal{D}_1 = \{\emptyset, \text{🍎}\}, \mathcal{D}_2 = \{\emptyset, \text{🍎🍎}\}, \mathcal{D}_3 = \{\emptyset, \text{🍎🍎🍎}\}, \ldots$$

So

$$D(x) = \prod_{m \geq 1} (1 + x^m)$$

Integer partition – odd parts and distinct parts

There are 6 partitions of 8 into distinct parts, 6 into odd parts. Coincidence?

8 distinct parts	4+1+1+1+1
7+1 distinct parts, odd parts	3+3+2
6+2 distinct parts	3+3+1+1 odd parts
6+1+1	3+2+2+1
5+3 distinct parts, odd parts	3+2+1+1+1
5+2+1 distinct parts	3+1+1+1+1+1 odd parts
5+1+1+1 odd parts	2+2+2+2
4+4	2+2+2+1+1
4+3+1 distinct parts	2+2+1+1+1+1
4+2+2	2+1+1+1+1+1+1
4+2+1+1	1+1+1+1+1+1+1+1 odd parts

Figure 8.15: The partitions of 8, noting those into distinct parts and those into odd parts.

Integer partition – odd parts and distinct parts

Theorem 8.16

For all $n \in \mathbb{N}$, the number of partitions of n into distinct parts equal to the number of partitions into odd parts.

$$
\begin{aligned}
D(x) &= \prod_{n=1}^{\infty}(1+x^n) = \prod_{n=1}^{\infty}\frac{1-x^{2n}}{1-x^n} = \frac{\prod_{n=1}^{\infty}(1-x^{2n})}{\prod_{n=1}^{\infty}(1-x^n)} \\
&= \frac{\prod_{n=1}^{\infty}(1-x^{2n})}{\prod_{n=1}^{\infty}(1-x^{2n-1})\prod_{n=1}^{\infty}(1-x^{2n})} = \prod_{n=1}^{\infty}\frac{1}{1-x^{2n-1}} \\
&= O(x).
\end{aligned}
$$

Challenge Find a proof without using GF. Or read this one.

Exponential generating functions

What is a EGF (exponential generating function)

Given an infinite sequence $\sigma = (a_0, a_1, \dots)$, we associate it with a "function" $F(x)$ written as

$$F(x) = \sum_{n \ge 0} \frac{a_n}{n!} x^n,$$

called the exponential generating function of σ.

Warning ⚠

Again $F(x)$ is not a function. We do not care if the sum converges.

There are EGF that does not correspond to any function, e.g.,

$$\sum_{n \ge 0} \frac{(n!)^2}{n!} x^n.$$

Example – strings

EGF is usually used for labeled structures. For example, let

$$\mathcal{A} = \{\emptyset, \{\text{🍎}_1\}, \{\text{🍎}_1, \text{🍎}_2\}, \{\text{🍎}_1, \text{🍎}_2, \text{🍎}_3\}, \dots\}$$

i.e., strings consist of only 🍎 (bags of labeled 🍎). The number of such strings of size n is $a_n = 1$. Thus the set $\mathcal{A}$ has EGF

$$A(x) = \sum_{n \ge 1} \frac{1}{n!} x^n = e^x$$

EGF and combinatorics

Combinatorial product of labeled structures

For sets $\mathcal{A}$ and $\mathcal{B}$ of labeled structures, let $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$.

Let $A(x) = a_n x^n/n!$, $B(x) = b_n x^n/n!$. Then

$$C(x) = A(x) + B(x) = \sum_{n \geq 0} \frac{a_n + b_n}{n!} x^n.$$

because there are $a_n + b_n$ objects in $\mathcal{C}$ of size n.

Example – Odd and even number of apples

If $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ with

$$\mathcal{A} = \{\emptyset, \{🍎1, 🍎2\}, \{🍎1, 🍎2, 🍎3, 🍎4\}, \dots\}$$
$$\mathcal{B} = \{\{🍎1\}, \{🍎1, 🍎2, 🍎3\}, \{🍎1, 🍎2, 🍎3, 🍎4, 🍎5\}, \dots\}$$

Then the EGF of $\mathcal{C}$ is

$$C(x) = A(x) + B(x) = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = e^x.$$

Quiz Can you get $C(x)$ directly?

Combining labeled structures

For these two bags of fruits

$$\{🍎1, 🍎2\}, \{🍊1, 🍊2\}$$

there are $\binom{4}{2} = 6$ ways to relabel them

$$\{🍎1, 🍎2, 🍊3, 🍊4\}, \{🍎1, 🍎3, 🍊2, 🍊4\}, \{🍎1, 🍎4, 🍊2, 🍊3\},$$
$$\{🍎2, 🍎3, 🍊1, 🍊4\}, \{🍎2, 🍎4, 🍊1, 🍊3\}, \{🍎3, 🍎4, 🍊1, 🍊2\}$$

This is to say, there are $6 = \binom{4}{2}$ strings of length 4 that consist of two 🍎 and two 🍊.

When two labeled structures are combined, relabeling is needed to avoid duplicate labels. If the two structures are of size k and $n - k$, then there are $\binom{n}{k}$ ways to relabel while keeping the relative order of labels in each of the two.

Combinatorial product of labeled structures

For sets $\mathcal{A}$ and $\mathcal{B}$ of labeled structures, let $\mathcal{C} = \mathcal{A} \star \mathcal{B}$ be the set of structures combining $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$ through relabeling.

Let $A(x) = a_n x^n/n!$, $B(x) = b_n x^n/n!$, $C(x) = c_n x^n/n!$ be the EGF of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$

Then

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$$

because there are a_k ways to choose α, b_{n-k} ways to choose β and $\binom{n}{k}$ ways to relabel. In other words,

$$C(x) = \sum_{n\geq 0} c_n x^n = \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right) x^n = A(x)B(x).$$

Examples of strings

Binary strings

The number of strings of length n on alphabet {🍎, 🍌}, i.e., 2^n. So the EGF is

$$\sum_{n\geq 0} \frac{2^n}{n!} x^n = \sum_{n\geq 0} \frac{1}{n!}(2x)^n = e^{2x}.$$

Another way – Let $\mathcal{A}, \mathcal{B}$ be the sets of strings of of {🍎}, {🍌}, {🍊}, or bags of fruits labeled by $1, 2, 3, \ldots$

$\mathcal{A} = \{\emptyset$, {🍎1}, {🍎1, 🍎2}, {🍎1, 🍎2, 🍎3}, … $\}$

$\mathcal{B} = \{\emptyset$, {🍌1}, {🍌1, 🍌2}, {🍌1, 🍌2, 🍌3}, … $\}$

Then $\mathcal{A} \star \mathcal{B}$ gives the set of binary strings. For example,

$\emptyset$ + {🍌1, 🍌2} → {🍌1, 🍌2}

{🍌1} + {🍎1} → {🍎1, 🍌2}, {🍎2, 🍌1}

{🍎1, 🍎2} + $\emptyset$ → {🍎1, 🍎2}

Binary strings

Let

$\mathcal{A} = \{\emptyset$, {🍎1}, {🍎1, 🍎2}, {🍎1, 🍎2, 🍎3}, … $\}$

$\mathcal{B} = \{\emptyset$, {🍌1}, {🍌1, 🍌2}, {🍌1, 🍌2, 🍌3}, … $\}$

$\mathcal{C} = \mathcal{A} \star \mathcal{B}$

The exponential generating functions of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are

$$A(x) = B(x) = \sum_{n\geq 0} \frac{1}{n!} x^n = e^x, \qquad C(x) = A(x)B(x)$$

Then the magic happens

$$[x^n]C(x) = [x^n]A(x)B(x) = [x^n]e^{3x} = [x^n]\sum_{n\geq 0} \frac{(3x)^n}{n!} = \frac{3^n}{n!}.$$

Ternary strings without restriction

Problem

How many strings of length n on the alphabet $\{$🍎, 🍌, 🍊$\}$ have > 0 🍎 and > 0 🍌

Such strings forms the set $\mathcal{T} = \mathcal{A} \star \mathcal{B} \star \mathcal{C}$,

$$\mathcal{A} = \{\{\text{🍎1}\}, \{\text{🍎1}, \text{🍎2}\}, \{\text{🍎1}, \text{🍎2}, \text{🍎3}\}, \dots\}$$
$$\mathcal{B} = \{\{\text{🍌1}\}, \{\text{🍌1}, \text{🍌2}\}, \{\text{🍌1}, \text{🍌2}, \text{🍌3}\}, \dots\}$$
$$\mathcal{C} = \{\emptyset, \{\text{🍊1}\}, \{\text{🍊1}, \text{🍊2}\}, \{\text{🍊1}, \text{🍊2}, \text{🍊3}\}, \dots\}$$

The EGFs are

$$T(x) = A(x)B(x)C(x) = (e^x - 1)^2 e^x = e^{3x} - 2e^{2x} + e^x.$$

So the answer is

$$n![x^n]\left(\sum_{n\ge 0} \frac{3^n x^n}{n!} - 2\sum_{n\ge 0} \frac{2^n x^n}{n!} + \sum_{n\ge 0} \frac{x^n}{n!}\right) = 3^n - 2^{n+1} + 1.$$

Ternary strings with restriction

What if the number of apple needs to be even? Let

$$\mathcal{A} = \{\emptyset, \{\text{🍎1}, \text{🍎2}\}, \{\text{🍎1}, \text{🍎2}, \text{🍎3}, \text{🍎4}\}, \dots\}$$
$$\mathcal{B} = \{\emptyset, \{\text{🍌1}\}, \{\text{🍌1}, \text{🍌2}\}, \{\text{🍌1}, \text{🍌2}, \text{🍌3}\}, \dots\}$$
$$\mathcal{C} = \{\emptyset, \{\text{🍊1}\}, \{\text{🍊1}, \text{🍊2}\}, \{\text{🍊1}, \text{🍊2}, \text{🍊3}\}, \dots\}$$

The exponential generating functions (EGF) of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are

$$A(x) = \sum_{n\ge 0} \frac{x^{2n}}{(2n)!} = ?\frac{e^x + e^{-x}}{2}, \quad B(x) = C(x) = e^x$$

Then the magic happens again

$$A(x)B(x)C(x) = \frac{e^x + e^{-x}}{2} e^{2x} = \frac{e^{3x} + e^x}{2} = \frac{1}{2}\left(\sum_{n\ge 0} \frac{3^n x^n}{n!} + \frac{x^n}{n!}\right).$$

So the answer is $n![x^n]A(x)B(x)C(x) = (3^n + 1)/2$.

A fruit password

Problem

We want an 8-fruit password consisting an even number of 🍎, > 0 🍌, ≤ 3 🍊 and unlimited number of 🍇 🍈 🍉 🍋 🍍 🍏 🍐.
How many such password are possible?

Such passwords form the set of $\mathcal{F} = \mathcal{A}_1 \star \mathcal{A}_2 \star \cdots \star \mathcal{A}_{10}$ with

$$\mathcal{A}_1 = \{\emptyset, \{\text{🍎}_1, \text{🍎}_2\}, \{\text{🍎}_1, \text{🍎}_2, \text{🍎}_3, \text{🍎}_4\}, \dots\}$$

$$\mathcal{A}_2 = \{\{\text{🍌}_1\}, \{\text{🍌}_1, \text{🍌}_2\}, \{\text{🍌}_1, \text{🍌}_2, \text{🍌}_3\}, \dots\}$$

$$\mathcal{A}_3 = \{\emptyset, \{\text{🍊}_1\}, \{\text{🍊}_1, \text{🍊}_2\}, \{\text{🍊}_1, \text{🍊}_2, \text{🍊}_3\}\}$$

$$\mathcal{A}_4 = \{\emptyset, \{\text{🍇}_1\}, \{\text{🍇}_1, \text{🍇}_2\}, \{\text{🍇}_1, \text{🍇}_2, \text{🍇}_3\}, \dots\}, \dots$$

$$\mathcal{A}_{10} = \{\emptyset, \{\text{🍐}_1\}, \{\text{🍐}_1, \text{🍐}_2\}, \{\text{🍐}_1, \text{🍐}_2, \text{🍐}_3\}, \dots\},$$

Thus the EGF of $\mathcal{F}$ is

$$F(x) = \frac{e^x + e^{-x}}{2}(e^x - 1)\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!}\right) e^{7x}$$

So the answer is $8![x^8]F(x) = 33847837$.

Appendix

Self-study guide (for people who missed the class)

- Read Textbook 8.5, 8.6.
- Recommended exercises Have a quick look of
  - Textbook 8.9, 20–28 (Some solutions here)