8 – Recurrence Equations

Combinatorics 1M020

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Introduction

Fibonacci sequence

The Fibonacci sequence is defined by

$$f(0) = 0, \quad f(1) = 1, \quad f(n) = f(n-1) + f(n-2) \quad (n \geq 2).$$

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First few terms are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...$$

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It appears in nature quite often

21 (blue) and 13 (aqua) spirals

Fibonacci sequence – the ratio

The ratio of $f(n+1)/f(n)$ seems to converges

$1/1 = 1.0000000000$	$89/55 = 1.6181818182$
$2/1 = 2.0000000000$	$144/89 = 1.6179775281$
$3/2 = 1.5000000000$	$233/144 = 1.6180555556$
$5/3 = 1.6666666667$	$377/233 = 1.6180257511$
$8/5 = 1.6000000000$	$610/377 = 1.6180371353$
$13/8 = 1.6250000000$	$987/610 = 1.6180327869$
$21/13 = 1.6153846154$	$1597/987 = 1.6180344478$
$34/21 = 1.6190476190$	$2584/1597 = 1.6180338134$
$55/34 = 1.6176470588$	$4181/2584 = 1.6180340557$

Fibonacci sequence – the ratio

A closed form

$$f(n) = \frac{1}{\sqrt{5}} \left(\varphi^n + \psi^n\right)$$

with

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.6180339887, \quad \psi = \frac{1-\sqrt{5}}{2} \approx -0.6180339887.$$

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$$\frac{f(n+1)}{f(n)} \approx \varphi.$$

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How can we get formula like this? This is what we will learn.

Ternary string – revisit

A ternary string of alphabet {🍎, 🍌, 🍊} is good if there's no 🍊 followed by 🍎. Example

- 🍎🍎🍌🍊🍌🍌🍎 – good
- 🍊🍎🍌🍌🍎🍎🍊🍌🍊🍊 – bad

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Then

$$t(n) = 3t(n-1) - t(n-2).$$

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Then

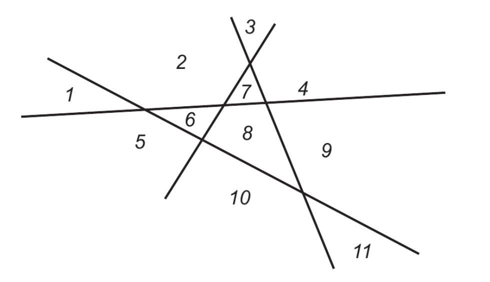
$$t(n) = 3t(n-1) - t(n-2).$$

Easy to check

$$t(n) = \frac{1}{10}\left(3\sqrt{5}+5\right)\left(\frac{3+\sqrt{5}}{2}\right)^n + \frac{1}{10}\left(5-3\sqrt{5}\right)\left(\frac{3-\sqrt{5}}{2}\right)^n$$

How can we get this?

Lines and areas–Recursion



Let n be the number of lines and $r(n)$ be the number of regions. Then

$$r(n) = n + r(n-1)$$

Linear Recurrence Equations

Definitions

A sequence $(a_n, n \geq 0)$ satisfies a linear recurrence if

$$c_0 a_{n+k} + c_1 a_{n+k-1} + c_2 a_{n+k-2} + \cdots + c_k a_n = g(n),$$

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The recursion is homogeneous if $g(n)$ is always 0. Example

$$f(n+2) - f(n+1) - f(n) = 0.$$

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Otherwise it is nonhomogeneous. Example

$$r(n+1) - r(n) = n + 1.$$

Advancement operator

Let $Af(n) = f(n+1)$ and $A^p f(n) = f(n+p)$.

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The recurrence

$$c_0 f(n+k) + c_1 f(n+k-1) + c_2 f(n+k-2) + \cdots + c_k f(n) = g(n),$$

can be written as

$$p(A)f = (c_0 A^k + c_1 A^{k-1} + c_2 A^{k-2} + \cdots + c_k)f = g.$$

The roots of $p(A)$

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Fact $p(A)$ has k non-zero roots $r_1, \dots, r_k \in \mathbb{C}$, i.e.,

$$p(A) = (A - r_1)(A - r_2)(A - r_3) \dots (A - r_k),$$

and $r_1 \neq 0, \dots, r_k \neq 0$.

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Example

Let $\mathbb{i} = \sqrt{-1}$. Then

$$A^2 + 1 = (A - \mathbb{i})^2$$

Not true if restricted to real solutions!

Properties of Advancement operator (1)

While

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we also have

$$\begin{aligned}(A+3)(A-2)f(n) &= (A+3)(f(n+1) - 2f(n)) \\ &= (f(n+2) - 2f(n+1)) + 3(f(n+1) - 2f(n)) \\ &= f(n+2) + f(n+1) - 6f(n) \\ &= (A^2 + A - 6)f(n)\end{aligned}$$

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Fact If $p(A) = q(A)$ as polynomial, then $p(A)f = q(A)f$.

Properties of Advancement operator (2)

Fact We have

$$p(A)(f_1(n) + f_2(n)) = p(A)f_1(n) + p(A)f_2(n)$$

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An example –

$$\begin{aligned}
&(A-2)(f_1(n) + f_2(n)) \\
&= A(f_1(n) + f_2(n)) - 2(f_1(n) + f_2(n)) \\
&= (f_1(n+1) + f_2(n+1)) - 2(f_1(n) + f_2(n)) \\
&= (A-2)f_1(n) + (A-2)f_2(n)
\end{aligned}$$

What if $c_k = 0$?

We assume that $c_k \neq 0$, in the linear recurrence

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So the solution for $p(A)h = g$ is $h(n) = h^*(n-2)$.

Solving advancement operator equations – homogeneous

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Quiz What is the solution for

$$(A+3)f(n)=0$$

First example

Example 9.9 – Find all solutions for

$$p(A)f = (A^2 + A - 6)f = (A+3)(A-2)f = 0$$

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If $(A-2)f_1 = 0$ then $(A+3)(A-2)f_1 = 0$.

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For example, $f_1(n) = 2^n$,

$$p(A)2^n = 2^{n+2} + 2^{n+1} - 6 \times 2^n = 2^n(4 + 2 - 6) = 0$$

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If $(A+3)f_2 = 0$ then $(A-2)(A+3)f_2 = 0$.

For example, $f_2(n) = (-3)^n$,

$$\begin{aligned} p(A)(-3)^n &= (-3)^{n+2} + (-3)^{n+1} - 6 \times (-3)^n \\ &= (-3)^n(9 - 3 - 6) = 0 \end{aligned}$$

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$$p(A)f = (A^2 + A - 6)f = (A+3)(A-2)f = 0$$

Together $c_1 2^n + c_2(-3)^n$ are also solutions, since

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Fact $c_1 2^n + c_2(-3)^n$ are all the solutions.
Fact If $p(A)$ is of degree k, then the general solution for $p(A)f = g$ has k parameters.

Example – Ternary strings

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all the solutions are like

$$t(n) = c_1 \left(\frac{3+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{3-\sqrt{5}}{2}\right)^n.$$

Example – Ternary strings

Since $t(0) = 1$ and $t(1) = 3$,

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Solving this

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Solving this

$$c_1 = \frac{1}{10}\left(3\sqrt{5}+5\right), \qquad c_2 = \frac{1}{10}\left(5-3\sqrt{5}\right)$$

and

$$t(n) = \frac{1}{10}\left(3\sqrt{5}+5\right)\left(\frac{3+\sqrt{5}}{2}\right)^n + \frac{1}{10}\left(5-3\sqrt{5}\right)\left(\frac{3-\sqrt{5}}{2}\right)^n$$

The distinct roots case

Theorem 9.21

Assume that for disintct $r_1, r_2, \dots r_k$,

$$p(A) = (A - r_1)(A - r_2) \dots (A - r_k).$$

Then every solution of $p(A)f = 0$ has the form

$$f(n) = c_1 r_1^n + c_2 r_2^n + \cdots + c_k r_k^n$$

Example – Repeated roots

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What about $c_2 n 2^n$?

$$\begin{aligned}(A-2)^2(c_2 n 2^n) &= (A-2)(c_2(n+1)2^{n+1} - 2c_2 n 2^n) \\ &= (A-2)(c_2 2^{n+1}) \\ &= c_2 2^{n+2} - 2c_2 2^{n+1} = 0\end{aligned}$$

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So the general solution is

$$f(n) = c_1 2^n + c_2 n 2^n$$

Example – Repeated roots

What is the solution for

$$(A+5)(A-1)^3 f = 0$$

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Quiz Can you guess another form of solution?

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Quiz Can you guess another form of solution? Answer $c_4 n^2$.

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Quiz Can you guess another form of solution? Answer $c_4 n^2$. So the general solution is

$$f(n) = c_1(-5)^n + c_2 + c_3 n + c_4 n^2.$$

Repated roots

Lemma 9.22

Let $k \geq 1$. Then general soltuion of

$$(A - r)^k f = 0$$

has the form

$$f(n) = c_1 r^n + c_2 n r^n + \cdots + c_k n^{k-1} r^n$$

Solving advancement operator equations – nonhomogeneous

Example – Nonhomogenous

Problem

What is the general solution for

$$p(A)f = (A+2)(A-6)f = 3^n$$

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Observation If $p(A)f_2 = 3^n$, then $p(A)(f_1 + f_2) = 3^n$.

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Observation If $p(A)f_2 = 3^n$, then $p(A)(f_1 + f_2) = 3^n$.

Punch line It is enough to find one particular solution f_2!

Example – Nonhomogenous – Finding a particular solution

Problem

Can you find any one solution for

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$$p(A)d3^n = -5d3^{n+1}$$

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And the general solution is

$$f(n) = f_1(n) + f_2(n) = c_1(-2)^n + c_2 6^n - \frac{1}{15}3^n$$

Nonhomogenous – Recipe

We want to solve problems like

$$p(A)f = g.$$

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Then the general solution is

$$f(n) = f_1(n) + f_2(n)$$

Example – Lines and areas

Problem

What is the general solution for

$$r(n+1) - r(n) = n+1 \qquad \longleftrightarrow \qquad (A-1)r = n+1$$

The general solution for homogeneous version is $f_1(n) = c_1$.

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$$(A-1)(d_1 n^2 + d_2 n) = 2d_1 n + d_1 + d_2.$$

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$$(A-1)(d_1 n^2 + d_2 n) = 2d_1 n + d_1 + d_2.$$

So $f_2(n) = n^2/2 + n/2$ is a solution. And the general solution is

$$f(n) = f_1(n) + f_2(n) = c_1 + \frac{n^2 + n}{2}$$

Using generating functions to solve recurrences

Example 9.24

We want to solve the linear recurrence

$$r_n + r_{n-1} - 6r_{n-2} = 0,$$

with $r_0 = 1$ and $r_1 = 3$.

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$$\begin{aligned} f(x) &= r_0 + r_1 x + r_2 x^2 + \ldots \\ xf(x) &= 0 + r_0 x + r_1 x^2 + r_2 x^3 + \ldots \\ -6x^2 f(x) &= 0 + 0 - 6r_0 x^2 - 6r_1 x^3 - 6r_2 x^4 + \ldots \end{aligned}$$

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Summing over the three equations, we have

$$(1 + x - 6x^2) f(x) = 1 + 4x$$

Example 9.24 – with GF

Therefore

$$f(x) = \frac{1+4x}{1+x-6x^2} = \frac{1+4x}{(1-2x)(1+3x)}$$

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$$f(x) = \frac{1+4x}{1+x-6x^2} = \frac{1+4x}{(1-2x)(1+3x)}$$

We can always turn this into partial sums

$$f(x) = \frac{6}{5}\frac{1}{1-2x} - \frac{1}{5}\frac{1}{1+3x} = \frac{6}{5}\sum_{n\geq 0}(2x)^n - \frac{1}{5}\sum_{n\geq 0}(-3x)^n$$

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Thus

$$r_n = [x^n]f(x) = \frac{6}{5}2^n - \frac{1}{5}(-3)^n.$$

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$$r_n - r_{n-1} - 2r_{n-2} = 2^n,$$

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Summing over $n \geq 2$,

$$\sum_{n\geq 2} r_n x^n - \sum_{n\geq 2} r_{n-1} x^n - 2\sum_{n\geq 2} r_{n-2} x^n = \sum_{n\geq 2} 2^n x^n$$

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Let $R(x) = \sum_{n\geq 0} r_n x^n$, then

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Let $R(x) = \sum_{n\geq 0} r_n x^n$, then

$$R(x) - (2+x) - (xR(x) - 2x) - 2x^2 R(x) = \frac{1}{1-2x} - (1+2x)$$

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$$R(x) - (2+x) - (xR(x) - 2x) - 2x^2 R(x) = \frac{1}{1-2x} - (1+2x)$$

In other words

$$R(x) = \frac{6x^2 - 5x + 2}{(1-2x)(1-2x)(1+x)}$$

Example 9.25 – Nonhomogenous case

Again, we can turn

$$R(x) = \frac{6x^2 - 5x + 2}{(1 - 2x)^2(1 + x)}$$

into partial fractions

$$R(x) = -\frac{1}{9(1 - 2x)} + \frac{2}{3(1 - 2x)^2} + \frac{13}{9(1 + x)}$$

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By Newton's binomial theorem

$$\frac{1}{(1-2x)^2} = \sum_{n\geq 0}\binom{-2}{n}(-2x)^n = \sum_{n\geq 0}(n+1)(2x)^n$$

Example 9.25 – Nonhomogenous case

Therefore

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implies that

$$-\frac{1}{9}\sum_{n\geq 0}(2x)^n + \frac{2}{3}\sum_{n\geq 0}(n+1)(2x)^n + \frac{13}{9}\sum_{n\geq 0}(-x)^n$$

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and

$$r_n = [x^n]R(x) = -\frac{1}{9}2^n + \frac{2}{3}(n+1)2^n + \frac{13}{9}(-1)^n$$

Pros and cons

Benefit of using GF – no need to guess a solution.

Disadvantage of using GF – often need to convert to partial fraction.

Use whatever method you want unless the problem specifically asks.

Appendix

Self-study guide (for people who missed the class)

- Watch online video lectures here.
- Read textbook chapter 9.1-9.4, 9.6
- Try exercises in textbook 9.9 (some solutions here)