# Combinatorics (1MA020) Mock Exam

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Student: ________________________________________________

## Grading method

- If the question asks for "finding a formula/number/generating function", then an answer **without** proof is **okay**. But if the answer is just slightly wrong, the problem will still get 0 points.
- If the question asks for "**proving**", then answer without a proof will get 0. Partially correct proof should get partial points.
- The final grade is rounded up -- i.e., 7.5 becomes 8.

## The problems

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### Problem 1

Throughout this problem, let $m \geq 0$, $n \geq 0$ be integers.

#### (a)

Let $S(n, m)$ be the number of surjections from $[n]$ to $[m]$. **Prove** the following formula

$$S(n, m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n$$

using the following Principle of Inclusion-Exclusion **(2p)**

Let $X$ be a set and let $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ be a family of properties. Then for each subset $S \subseteq [m]$, let $N(S)$ denote the number of elements of $X$ which satisfy property $P_i$ for all $i \in S$. Note that if $S = \emptyset$, then $N(S) = |X|$, as every element of $X$ satisfies every property in $S$ (which contains no actual properties).

**Theorem 7.7** (Principle of Inclusion-Exclusion). *The number of elements of $X$ which satisfy none of the properties in $\mathcal{P}$ is given by*

$$\sum_{S\subseteq[m]} (-1)^{|S|} N(S). \qquad (7.2.1)$$

## (b)

Let $\left\{ n \atop m \right\}$ be the number of ways to partition $[n]$ into $m$ nonempty subsets. (This is known as Stirling partition number.)

For example $\left\{ 3 \atop 2 \right\} = 3$ since there are only three ways to partition [3] into 2 nonempty subsets, i.e.,

```
{{1}, {2, 3}},   {{1, 2}, {3}},   {{1, 3}, {2}}
```

Find a formula for computing $\left\{ n \atop m \right\}$ using $S(n, m)$. (**1 p**)

## (c)

Let $B(n)$ be the number of ways to to partition $[n]$ into non-empty subsets. (This is known as Bell number.)

For example $B(3) = 5$, since there are only 5 ways to partition [3] into nonempty subsets, i.e.,

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{{1}, {2, 3}},   {{1, 2}, {3}},   {{1, 3}, {2}},   {{1}, {2}, {3}},   {{1, 2, 3}}
```

Find a formula for computing $B(n)$ using $\left\{ n \atop m \right\}$. (**1 p**)

## (d)

Prove that for all positive integers $n$ (**1 p**)

$$B(n+1) = \sum_{k=0}^{n} B(n-k) \binom{n}{k}$$

# Problem 2

The Fibonacci sequence satisfies the recurrence $f(n) = f(n-1) + f(n-2)$. In other words, $f$ satisfies

$$\left(A^2 - A - 1\right) f = 0 \tag{1}$$

where $A$ is the Advancement Operator, i.e., $A\, f(n) = f(n+1)$.

## (a)

Find the general solution for equation (1) (**2 p**)

## (b)

Find the particular solution for equation (1) which satisfies $f(0) = 0$ and $f(1) = 1$. (**1 p**)

## (c)

Find the general solution for the following equation (**2 p**)

$$\left(A^2 - A - 1\right) f = 2 \times 3^n$$

# The solutions

## Problem 1

### (a)

The following proof is copied from the textbook.

For positive integers $n$ and $m$, let $S(n, m)$ denote the number of surjections from $[n]$ to $[m]$. Note that $S(n, m) = 0$ when $n < m$. In this section, we apply the Inclusion-Exclusion formula to determine a formula for $S(n, m)$. We start by setting $X$ to be the set of all functions from $[n]$ to $[m]$. Then for each $f \in X$ and each $i = 1, 2, \ldots, m$, we say that $f$ satisfies property $P_i$ if $i$ is not in the range of $f$.

**Lemma 7.8.** *For each subset $S \subseteq [m]$, $N(S)$ depends only on $|S|$. In fact, if $|S| = k$, then*

$$N(S) = (m - k)^n.$$

The important part of the proof is

- Defining $P_i$ correctly.
- Get $N(S)$ correctly.

Each part gets 1 point.

### (b)

$$\left\{ {n \atop m} \right\} = \frac{1}{m!} S(n, m)$$

### (c)

$$B(n) = \sum_{m=0}^{n} \left\{ {n \atop m} \right\}$$

### (d)

Let $k + 1$ be the size of the part in the partition that contains $n + 1$. Besides $n + 1$, there are $\binom{n}{k}$ ways to choose the $k$ elements which go into the same part as $n + 1$. There are $n - k$ elements left and there are $B(n - k)$ ways to partition them. So for fixed $k$, there are $\binom{n}{k} B(n - k)$ ways to partition $[n + 1]$ such that $n + 1$ belongs to a part of size $k + 1$. Summing over $k$ from 0 to $n$ thus give $B(n + 1)$.

# Problem 2

## (a)

Since the roots of

$-1 - A + A^2 == 0$

are

$\{\{A \to \frac{1}{2}(1-\sqrt{5})\}, \{A \to \frac{1}{2}(1+\sqrt{5})\}\}$

the general solution is

$\left(\frac{1}{2}(1-\sqrt{5})\right)^n c_1 + \left(\frac{1}{2}(1+\sqrt{5})\right)^n c_2$

## (b)

We need to solve

$\{c_1 + c_2 == 0, \frac{1}{2}(1-\sqrt{5})\, c_1 + \frac{1}{2}(1+\sqrt{5})\, c_2 == 1\}$

This gives

$\{\{c_1 \to -\frac{1}{\sqrt{5}}, c_2 \to \frac{1}{\sqrt{5}}\}\}$

So the solution is

$-\frac{\left(\frac{1}{2}(1-\sqrt{5})\right)^n}{\sqrt{5}} + \frac{\left(\frac{1}{2}(1+\sqrt{5})\right)^n}{\sqrt{5}}$

## (c)

We only need to find a particular solution for

$$(A^2 - A - 1) f = 2 \times 3^n$$

We can try $d\,3^n$. The left hand side is

$5 \times 3^n d$

So we can simply take

$\{\{d \to \frac{2}{5}\}\}$

In other words, $\frac{2}{5} 3^n$ is a solution. And the general solution is thus

$\frac{2 \times 3^n}{5} + \left(\frac{1}{2}(1-\sqrt{5})\right)^n c_1 + \left(\frac{1}{2}(1+\sqrt{5})\right)^n c_2$