Inversions in split trees and conditional Galton–Watson trees

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The definition

Inversions in a permutation

- Let $\sigma_1, \dots, \sigma_n$ be a permutation of $\{1, \dots, n\}$.
- If $i < j$ and $\sigma_i > \sigma_j$, then the pair (σ_i, σ_j) is called an *inversion*.

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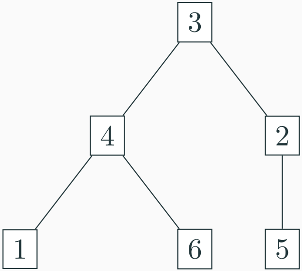
Inversions in a fixed tree

- Let T be a tree with node set V.
- Let λ be node labeling $\lambda : V \to \{1, \dots, |V|\}$.
- Define the number of *inversions*

$$I(T, \lambda) \stackrel{\text{def}}{=} \sum_{u<v} \mathbf{1}_{\lambda(u)>\lambda(v)}.$$



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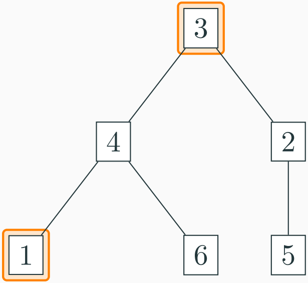

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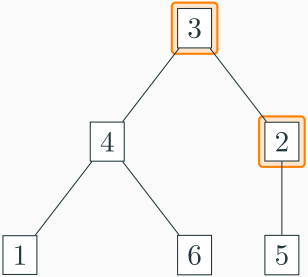

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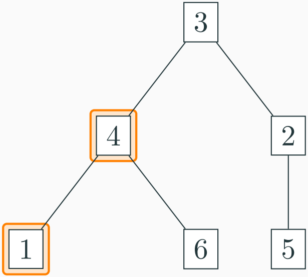

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- Panholzer and Seitz (2012) generalized this to conditional Galton–Watson trees.

Expectations

- Note that

$$\mathbb{E}\left[I(T)\right] = \sum_{u<v} \mathbb{E}\left[\mathbf{1}_{\lambda(u)>\lambda(v)}\right] = \frac{1}{2}\sum_{u<v} 1 \stackrel{\text{def}}{=} \frac{1}{2}\Upsilon(T).$$

- $\Upsilon(T)$ is also known as the *total path length*, since

$$\Upsilon(T) \stackrel{\text{def}}{=} \sum_{v} d(v),$$

where $d(v)$ is the depth of v.

Inversions in a sequence of trees

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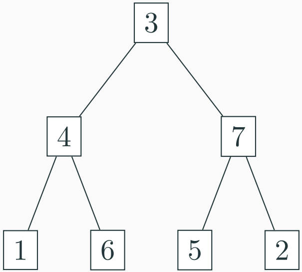
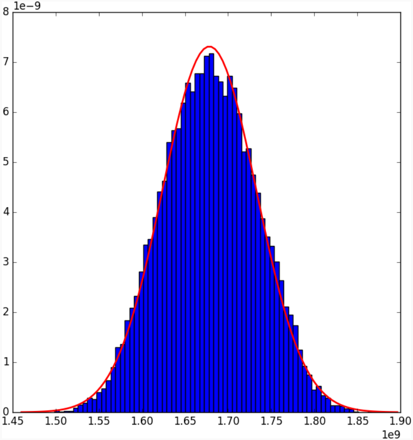


Simulations

- We did simulation on the complete binary tree of height 26
- Does the result suggest a central limit theorem?



The method of moments

- It is easy to see

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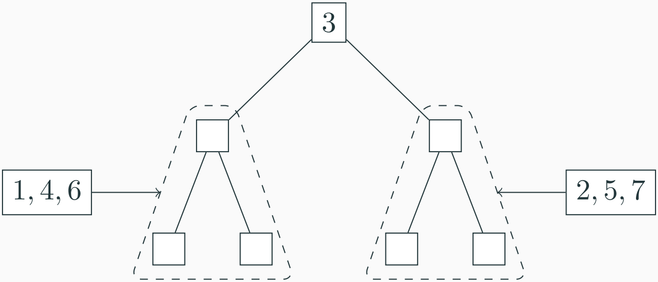
- So it cannot be a normal distribution!

A key observation

- Let Z_o be the number of inversions involving the root.
- Then Z_o and the numbers of inversions in the left subtree and right subtree are independent.
- Proof by conditioning on the labels that go the left and the right.



A key lemma

- Let z_v be the size of the subtree at v.
- Let Z_v be the number of inversions involving v and one of its descendants.

Lemma 1

Let T be a fixed tree. Then

$$I(T) \stackrel{\mathrm{d}}{=} \sum_{v \in V} Z_v,$$

where $\{Z_v\}_{v \in V}$ are independent random variables, and $Z_v \sim \mathrm{Unif}\{0, 1, \dots, z_v - 1\}$.

Cumulants

- The cumulant-generating function of a r.v. X is

$$K_X(t) = \log \mathbb{E}\left[e^{tX}\right].$$

- The cumulants $\varkappa_k(X)$ are defined by

$$K_X(t) = \sum_{k \geq 1} \varkappa_k(X) \frac{t^k}{k!}.$$

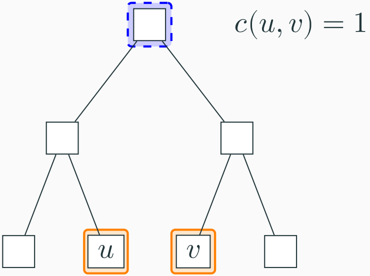
- If X is independent of Y, then

$$\varkappa_k(X+Y) = \varkappa_k(X) + \varkappa_k(Y).$$

- We can compute centralized-moments from cumulants.

k-total common ancestors

- For k nodes $v_1, \dots, v_k$, let $c(v_1, \dots, v_k)$ be the number of ancestors that they share.
- We define
$$\Upsilon_k(T) \stackrel{\text{def}}{=} \sum_{v_1,\dots,v_k} c(v_1, \dots, v_k).$$
- Note that $\Upsilon(T) = \Upsilon_1(T) - |V|$.



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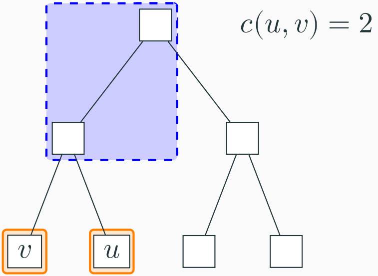

Cumulants for $I(T)$

Theorem 2

We have

$$\mathbb{E}\left[I(T)\right] = \varkappa_1(I(T)) = \frac{1}{2}(\Upsilon_1(T) - |V|),$$
$$\mathrm{Var}\left(I(T)\right) = \varkappa_2(I(T)) = \frac{1}{12}(\Upsilon_2(T) - |V|).$$

More generally, for $k \geq 1$,

$$\varkappa_{2k+1}(I(T)) = 0, \qquad \varkappa_{2k}(I(T)) = \frac{B_{2k}}{2k}(\Upsilon_{2k}(T) - |V|),$$

where B_k denotes the k-th Bernoulli number.

The condition for convergence

Theorem 3

Let T_n be a sequence of fixed trees on n nodes. Let

$$X_n = \frac{I(T_n) - \mathbb{E}\left[I(T_n)\right]}{\sqrt{\Upsilon_2(T_n)}}.$$

Assume that for all $k \geq 1$,

$$\frac{\Upsilon_{2k}(T_n)}{\Upsilon_2(T_n)^k} \to \zeta_{2k},$$

for some sequence (ζ_{2k}). Then there exists a unique X with

$$\varkappa_{2k-1}(X) = 0, \qquad \varkappa_{2k}(X) = \frac{B_{2k}}{2k}\zeta_{2k}, \qquad k \geq 1,$$

such that $X_n \xrightarrow{\mathrm{d}} X$.

Back to the Path

- Let $T_n = P_n$ (a path of length n),
$$\Upsilon_k(T_n) \sim \frac{1}{k+1} n^{k+1}.$$
- Thus
$$\frac{\Upsilon_{2k}(T_n)}{\Upsilon_2(T_n)^k} \to 0 \qquad (k \geq 2).$$
- So $(I(T_n) - \mathbb{E}[I(T_n)])/n$ converges to X with
$$\varkappa_k(X) = 0, \qquad (k \geq 3).$$
- Then X must be a normal distribution.

Back to complete binary trees

Theorem 4

Let $b \geq 2$ and let T_n be the complete b-ary tree of height m with n nodes. Then

$$X_n = \frac{I(T_n) - \mathbb{E}\left[I(T_n)\right]}{n} \xrightarrow{\mathrm{d}} \sum_{d \geq 0} \sum_{j=1}^{b^d} \frac{U_{d,j}}{b^d},$$

where $(U_{d,j})_{d \geq 0, j \geq 1}$ are independent $\mathrm{Unif}[-1/2, 1/2]$.

Inversions in split trees

Binary search trees (BST)

- BST is a computer data structure for storing "item" according to the order of their "keys".
- BST can be defined with a bijection to permutations.
- The average height of a BST of size n is $\alpha \ln n - \beta lnlnn$ [Reed, 2003].

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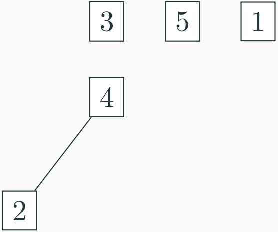

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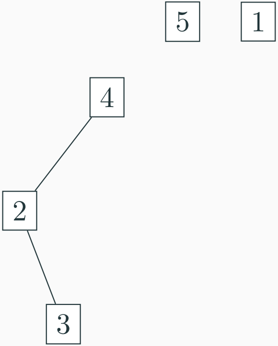

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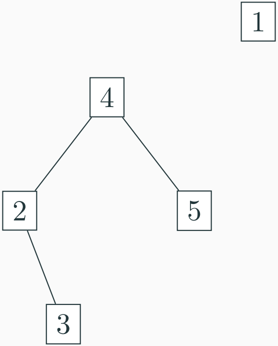

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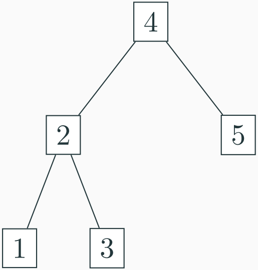
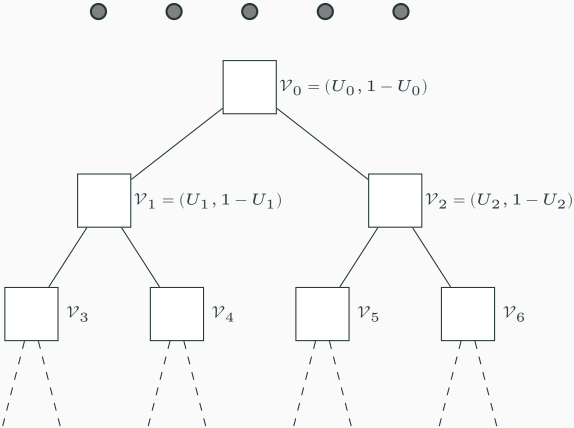


BST as a split tree i

- We can construct BST in another way.
- Consider an infinite binary tree.
- Each node is a "bucket" of size one.
- Each node is given a split vector $\mathcal{V} = (U, 1 - U)$ chosen independently.
- n balls come into the root one by one.
- When a bucket has more than one node, the extra goes to child nodes chosen at random according to $\mathcal{V}$.
- All empty buckets are removed in the end.

BST as a split tree ii



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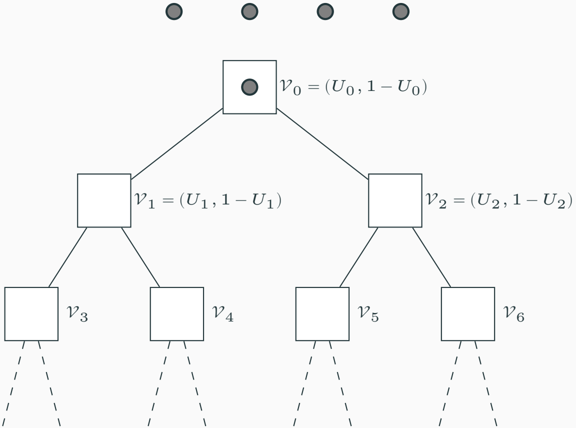

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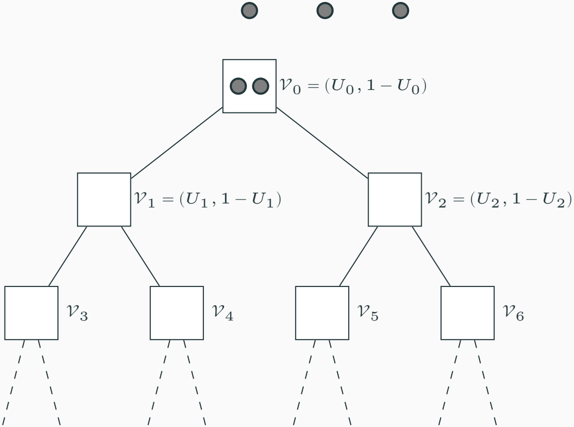

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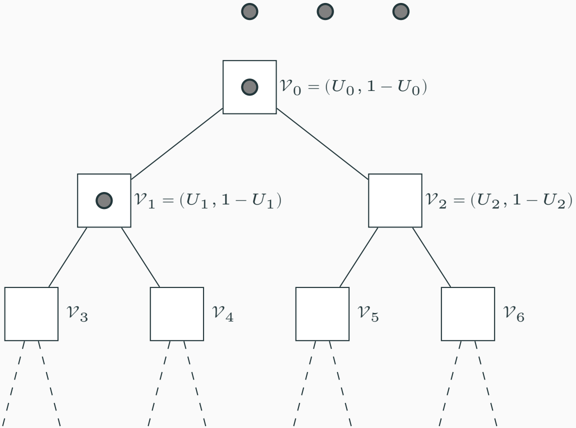

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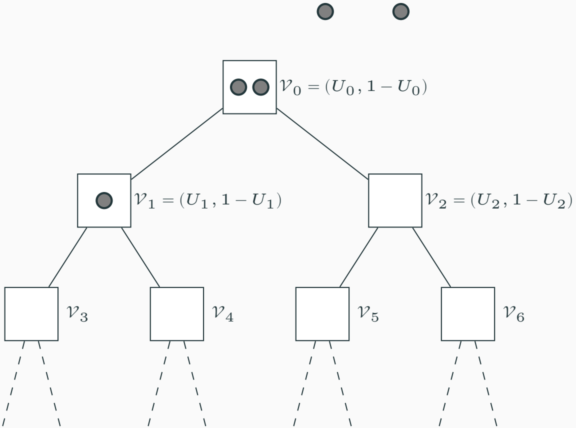

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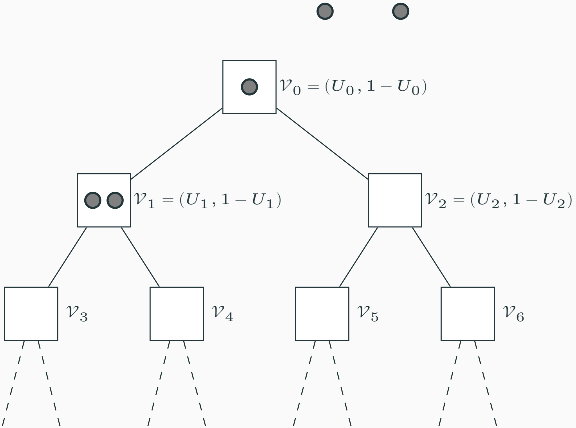

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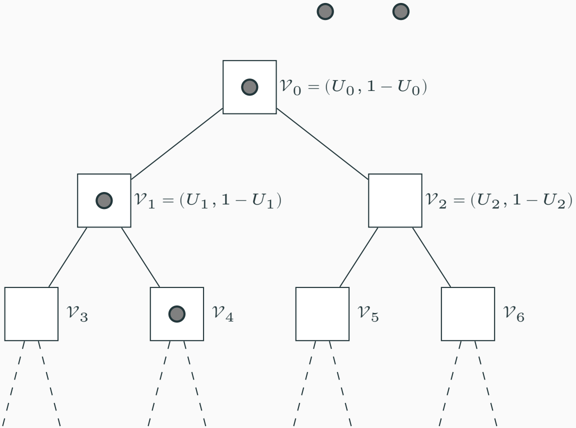

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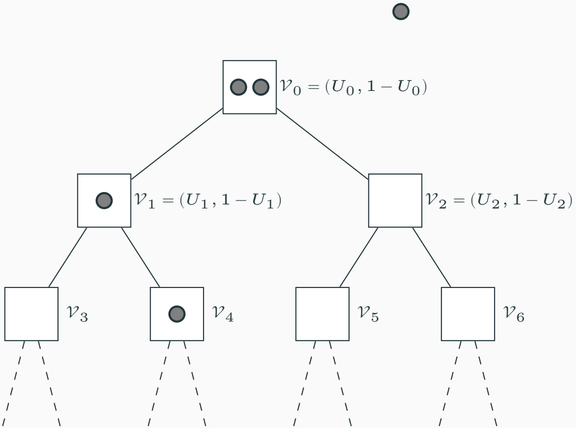

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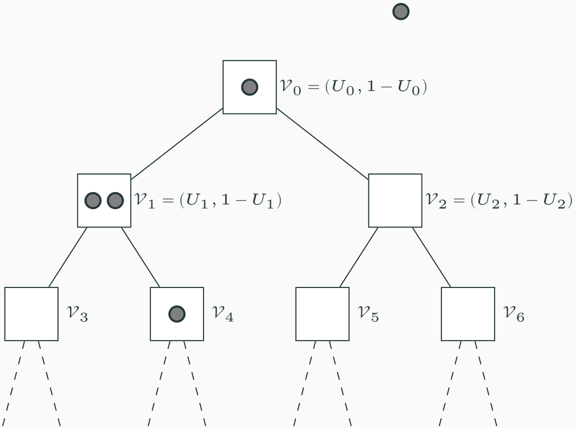

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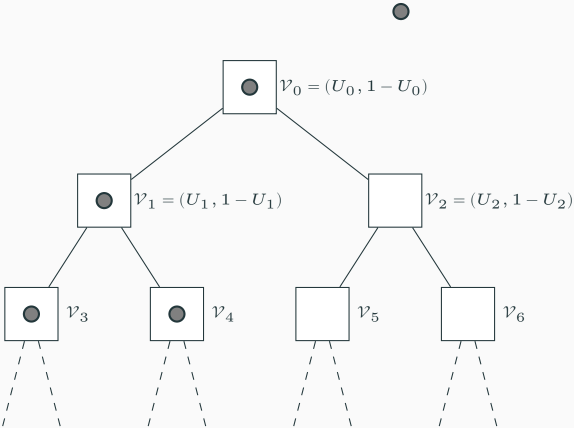

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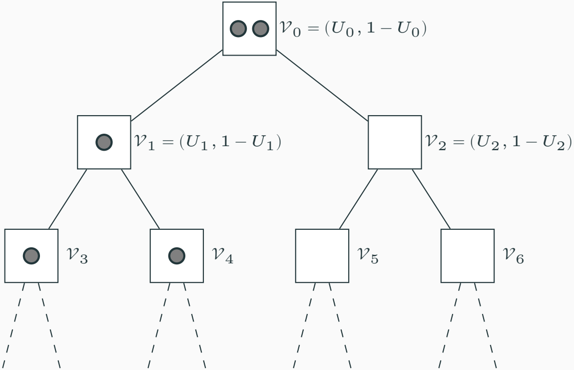

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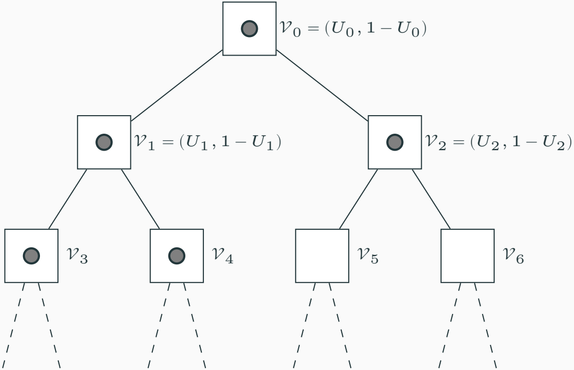
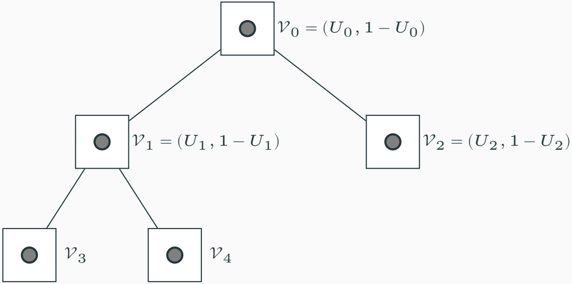


BST as a split tree ii



Split trees

- By choosing different:
  - inifnite trees
  - bucket sizes
  - distributions of split vector
- split trees encompasses:
  - binary search trees
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  - tries, etc.
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Inversions in split trees

- We first choose the split tree T_n with n balls.
- Then we randomly label the balls.
- We define $\widehat{I}(T_n)$ as the number of inversions for balls.
- We study

$$\hat{X}_n = \frac{\widehat{I}(T_n) - \mathbb{E}\left[\widehat{I}(T_n)\right]}{n}.$$

Results for split trees

Theorem 5

Let T_n be a b-ary split tree with bucket size s_0. Let $\mathcal{V} = (V_1, \dots, V_b)$ be a split vector. Let $\hat{X}$ be the unique solution for the fixed-point equation

$$\hat{X} \overset{\mathrm{d}}{=} \sum_{i=1}^{b} V_i \hat{X}^{(i)} + \sum_{j=1}^{s_0} U_j + \frac{s_0}{2} D(\mathcal{V}).$$

Then $\hat{X}_n \xrightarrow{\mathrm{d}} \hat{X}$.

- Proof by the contraction method.
- A similar result holds for labeling nodes instead of balls.

Inversions in conditional Galton-Watson trees

Galton-Watson trees

- A Galton–Watson tree starts with a root node.
- Each node in the tree is given a random number of child nodes.
- The numbers of children are independent with distribution ξ.

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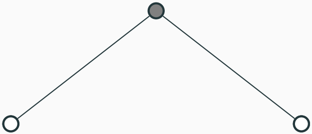

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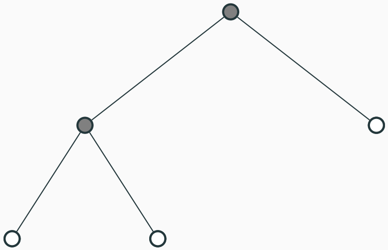

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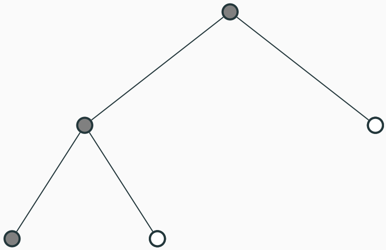

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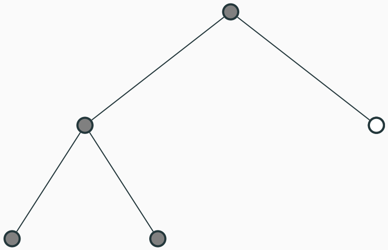

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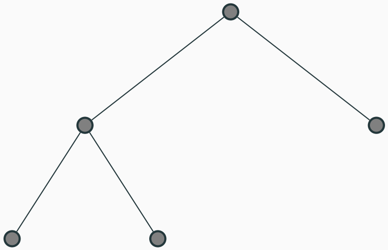
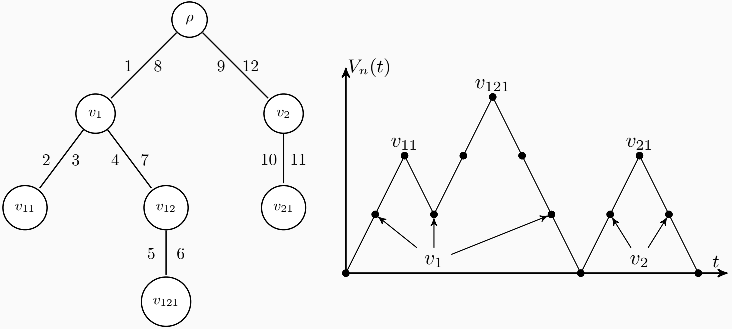


Conditional Galton–Watson

- A conditional Galton–Watson tree T_n is a Galton–Watson tree conditioned on having n nodes.
- It encompasses (uniform random)
  - plane trees
  - binary trees
  - b-ary trees
  - Cayley trees
- Very well studied, see, e.g., Janson (2012).

The depth-first walk on a conditional Galton–Watson tree



It is well-known that the depth-first walk on conditional GW trees converges to Brownian excursions [Aldous (1991a), Aldous (1991b), Aldous (1993), and Le Gall (2005)].

Let $e(t)$ be a Brownian excursion. Let

$$\eta \stackrel{\text{def}}{=} 4 \int_{0 \le s \le t \le 1} \min_{s \le u \le t} e(u) \mathrm{d}s \mathrm{d}t.$$

Theorem 6

Assume that $\mathbb{E}\left[\xi\right] = 1$, $\mathrm{Var}\left(\xi\right) = \sigma^2 \in (0, \infty)$, *and* $\mathbb{E}\left[e^{\alpha\xi}\right] < \infty$ *for some* $\alpha > 0$. *Then*

$$\frac{I(T_n) - \frac{1}{2}\Upsilon(T_n)}{n^{5/4}} \xrightarrow{\mathrm{d}} \frac{1}{\sqrt{12\sigma}} \sqrt{\eta}\, N(0,1).$$

Results for conditional GW trees ii

- Let
$$X_n = \frac{I(T_n) - \mathbb{E}[I(T_n)]}{n^{3/2}}.$$

- Then we can decompose
$$X_n = \frac{I(T_n) - \frac{1}{2}\Upsilon(T_n)}{n^{3/2}} + \frac{\Upsilon(T_n) - \mathbb{E}[\Upsilon(T_n)]}{2n^{3/2}}.$$

- Our result shows that the first term goes to zero.
- Aldous (1991b) showed that $\Upsilon(T_n)/n^{-3/2}$ converges to an Airy distribution.
- So X_n also converges to an Airy distribution.
- We recover result from Panholzer and Seitz, 2012.

Patterns on trees

- Let σ be a permutation of $\{1, \ldots, k\}$. Let

$$R_\sigma(T, \lambda) = \sum_{u_1 < \cdots < u_k} 1_{[\lambda(u_1, \ldots, u_k) = \sigma]}.$$

- Then $R_{21}(T, \lambda) = I(T, \lambda)$.
- Recently Albert, Holmgren, Johansson, and Skerman, 2018 studied $R_\sigma(T, \lambda)$ for complete binary trees and split trees.
- The spirit – most questions about permutations can be asked for trees.

Questions?

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