The graph structure of a deterministic automaton chosen at random

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k-out digraph

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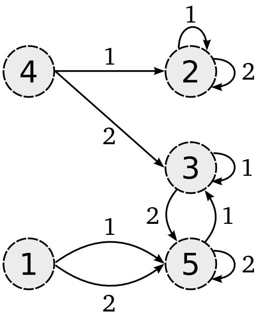

Figure 1 : A 2-out digraph with 5 vertices.

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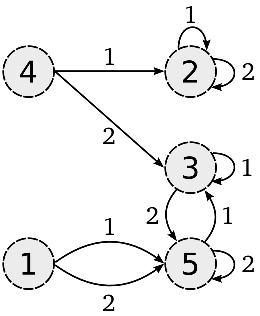

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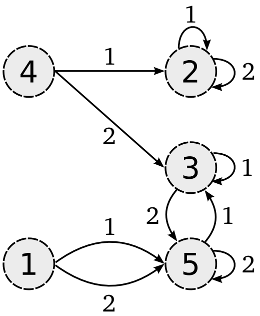

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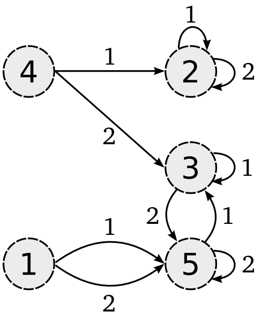
- Assume $k \geq 2$.



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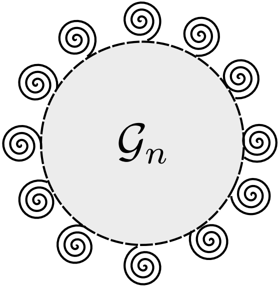

Figure 2 : Since $\nu_k > 1/2$, $\mathcal{G}_n$ is the largest SCC. We call $\mathcal{G}_n$ the giant.

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- In 2015, Addario-Berry, Balle, and Perarnau [1] proved that the diameter and the typical distance of $\mathcal{D}_{n,k}$, rescaled by $\log n$, converge in probability to constants.
- Also in 2014, Angluin and Chen [2] studied the mixing time of simple random walk on $\mathcal{D}_{n,k}$.

Our Result 1 — Cycles outside the giant

Let $\mathcal{G}_n^c \equiv [n] \setminus \mathcal{G}_n$. Let $\mathcal{D}_{n,k}[\mathcal{G}_n^c]$ be the sub-digraph induced by $\mathcal{G}_n^c$.

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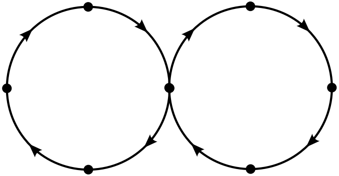
2. *Let C_n be the number of cycles in $\mathcal{D}_{n,k}[\mathcal{G}_n^c]$. Then*

$$C_n \xrightarrow{d} \mathrm{Poi}\left(\log \frac{1}{1 - k\mu_k}\right).$$

The intuition of Theorem 2

- When two cycles share vertices, they contain fewer vertices than arcs.



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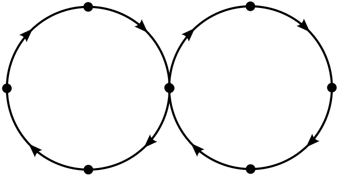

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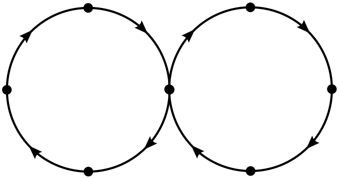

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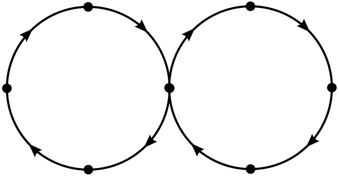
- The dependence between these indicators is very small.



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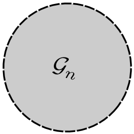

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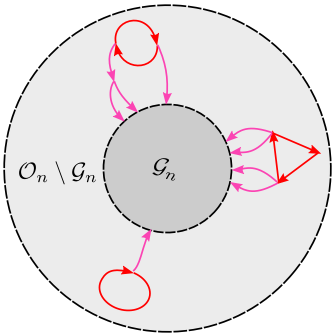

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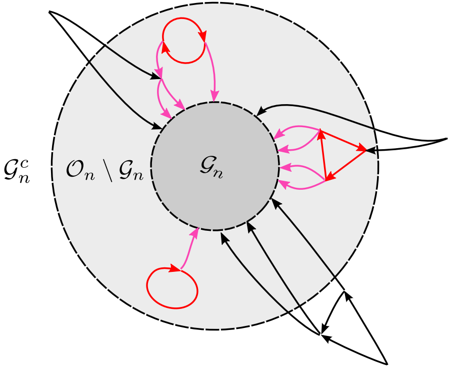

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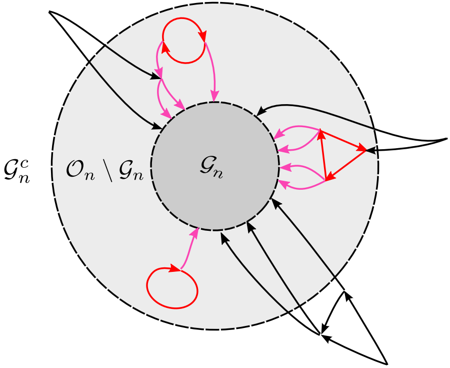

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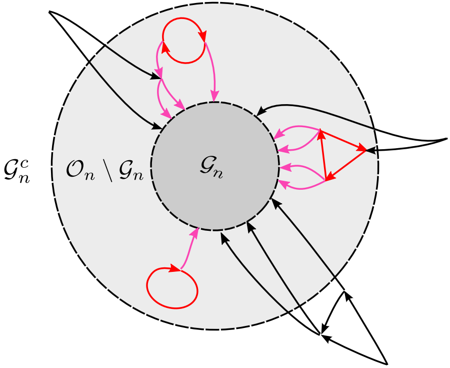
- We call $\mathcal{O}_n$ the one-in-core.



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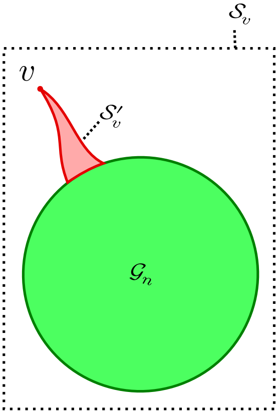

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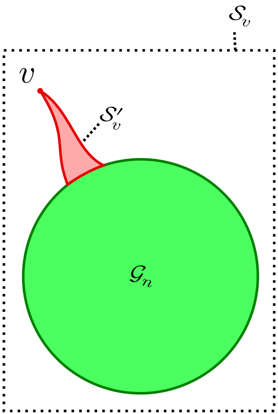

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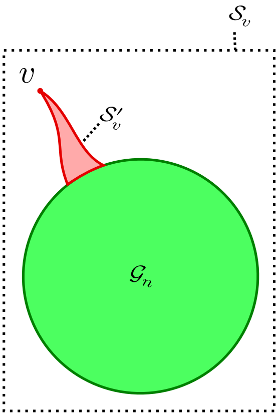

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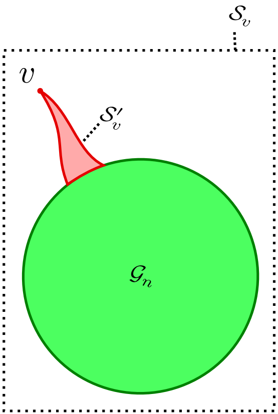

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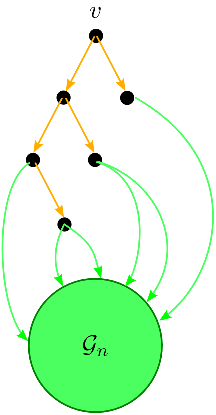

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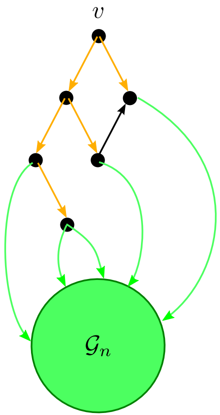

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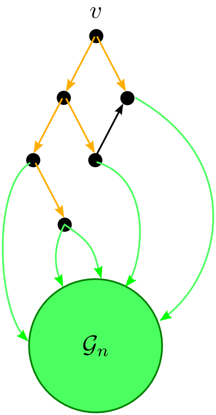

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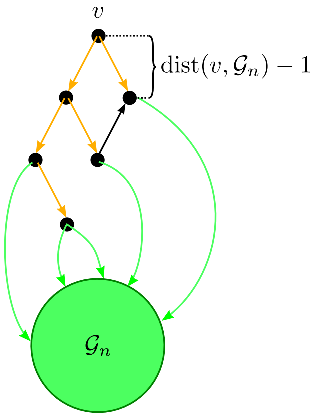

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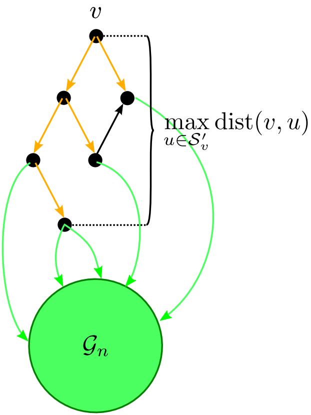

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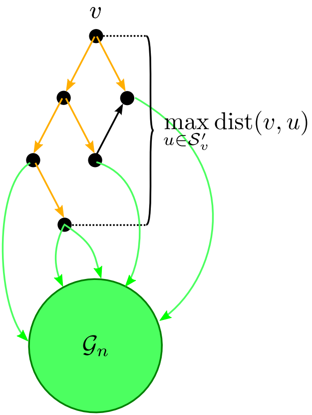

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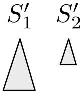

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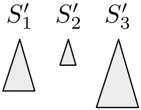

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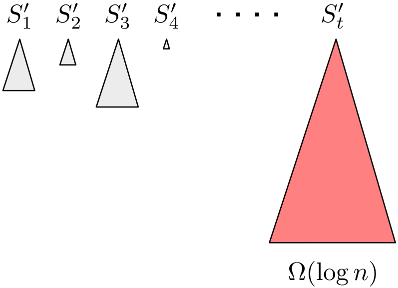

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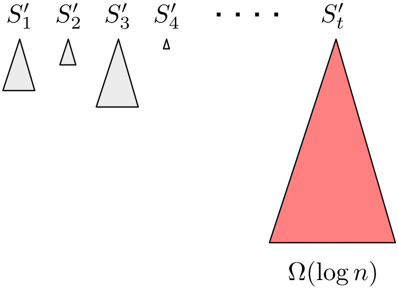

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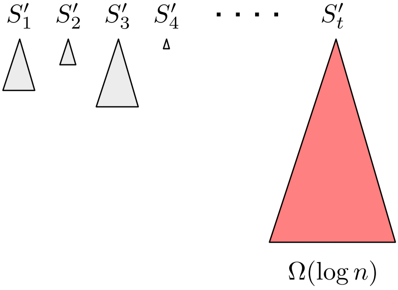

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Our result 4 – Strong connectivity

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Theorem 5 (Phase transition in strong connectivity)

1. *If* $k - \log n \to -\infty$, *then whp* $\mathcal{D}_{n,k}$ *is not strongly connected.*
2. *If* $k - \log n \to \infty$, *then whp* $\mathcal{D}_{n,k}$ *is strongly connected.*

Some open questions

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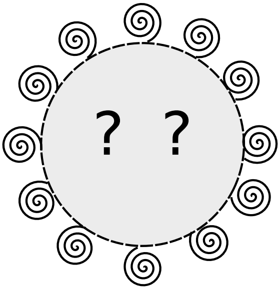
4. Generating a uniform random surjection from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$ for arbitrary $m = m(n)$.

Acknowledgment and Questions

1. The authors thank Laura Eslava, Hamed Hatami, Guillem Perarnau, Bruce Reed, Henning Sulzbach and Yelena Yuditsky for valuable comments on this work,



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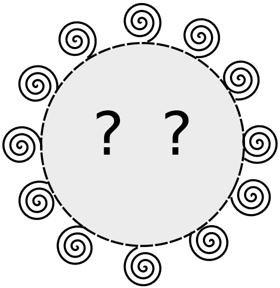

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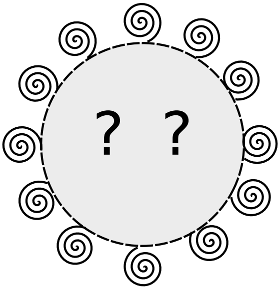

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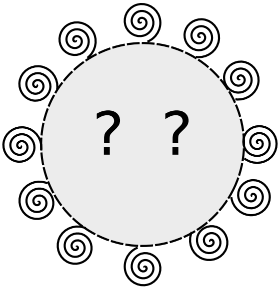
4. Thanks for listening!



References

[1] L. Addario-Berry, B. Balle, and G. Perarnau. Diameter and Stationary Distribution of Random r-out Digraphs. *ArXiv e-prints*, 2015.

[2] D. Angluin and D. Chen. Random walks on random uni-regular graphs. 2015.

[3] A. Carayol and C. Nicaud. Distribution of the number of accessible states in a random deterministic automaton. In *29th International Symposium on Theoretical Aspects of Computer Science*, volume 14 of *Leibniz International Proceedings in Informatics*, 194–205. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2012.

[4] A. A. Grusho. Limit distributions of certain characteristics of random automaton graphs. *Mathematical Notes of the Academy of Sciences of the USSR*, 14(1):633–637, 1973.